DIVERSIFYING INVESTMENTS AND MAXIMIZING SHARPE RATIO: A NOVEL QUBO FORMULATION

Luca Asproni | Data Reply
Christian Mattia | Intesa Sanpaolo

SPEAKERS

LUCA ASPRONI
REPLY DATA
Senior Quantum Computing Expert
l.asproni@reply.it

CHRISTIAN MATTIA
INTESA SANPAOLO
Quantum Optimization Specialist
christian.mattia@intesasanpaolo.com

AGENDA

- **Portfolio optimization: problem overview and data description**
- **Novel formulation: QUBO for maximization of diversification and Sharpe Ratio**
- **Results and conclusions**

PORTFOLIO OPTIMIZATION: PROBLEM OVERVIEW

PORTFOLIO OPTIMIZATION

Overview & Goals

- **Identify the optimal investments on a basket of assets** satisfying desired conditions on the expected return and portfolio risk
- **Provide a diversified portfolio**, allocating budget on multiple sectors

Design Details

- First component of objective function: **Sharpe Ratio maximization**

$$Sharpe\ Ratio = \frac{w^T \mu}{\sqrt{w^T \Sigma w}}$$

  - Σ variance-covariance matrix
  - $w_i \in [0,1]$ amount invested
  - on $i - th$ asset

- Second component: **diversification maximization** which makes the problem, in general, **non convex**, and thus **hard to solve with standard techniques**

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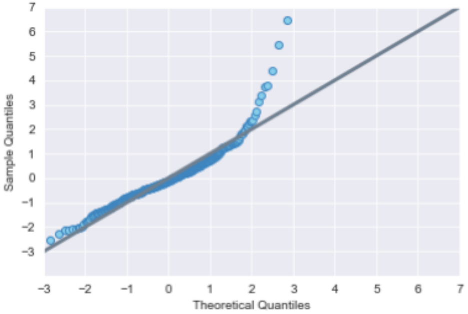
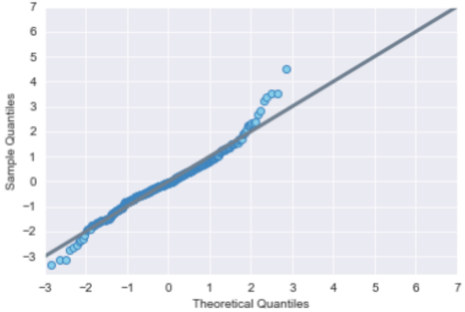
DATA DESCRIPTION



(a) Simple returns



(b) Log returns

- Historical S&P500 asset prices taken as daily adjusted closes from 2010 to 2020
- Returns and covariance matrix are annualized, i.e. their value is multiplied by 252 (assumed number of working days in a year)
- Historical returns are considered in the form of log-returns instead of simple returns, as suggested by Shapiro – Q-Q plot shown in figure as a qualitative comparison of the distributions

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NOVEL FORMULATION: QUBO FOR MAXIMIZATION OF DIVERSIFICATION AND SHARPE RATIO

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QUBO MODEL

- Quantum annealers minimize Ising Energy

$$H = \sum_{i} h_i S_i + \sum_{ij} J_{ij} S_i S_j$$

- Changing variables, the problem is equivalently written as a QUBO:

$$\min E(x|Q) = \sum_{i \leq j} x_i Q_{ij} x_j$$

Q | **Quadratic:**
the highest power of variables is x^2

U | **Unconstrained:**
no external constraints are applied

B | **Binary:**
variables are binary {0,1}

O | **Optimization:**
Minimization of an objective function (the energy)

STEP 1: REFORMULATION

Portfolio selection
(current model from literature)

$$\sum_i a_i q_i + \sum_i \sum_j b_{ij} q_i q_j$$

With:

$q_i \in \{0,1\}$, $a_i = \frac{\mu_i}{\sigma_i}$ and $b_{ij} = \rho_{ij}$, where ρ_{ij} is the correlation between asset i and asset j

→

Proxy formulation

$$\sum_{i=1}^{n} a_i \left(\sum_{l=0}^{p-1} d_l x_{il} \right) + \sum_{i=1}^{n} \sum_{j=i+1}^{n} b_{ij} \left(\sum_{l=0}^{p-1} d_l x_{il} \right) \left(\sum_{l=0}^{p-1} d_l x_{jl} \right)$$

With:

$x_i \in \{0,1\}$, $a_i = \frac{\mu_i}{\sigma_i}$, $b_{ij} = \rho_{ij}$, $d_l = \frac{2^l}{500}$ allocation discretization coefficient and $d_{p-1} = 1 - \sum_{l=0}^{p-2} d_l$, investment granularity $= 0.2\%$

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Proxy Formulation does not mathematically equate to the maximization of the Sharpe Ratio as it is defined in the classical problem!

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Classical model
(quadratic formulation)

$$\min y^T \Sigma y$$
$$s.t\ \mu^T y = 1, \qquad (y,k) \in Y$$

With:

$Y := \{y \in R^N, k \in R \mid k > 0, w = \frac{y}{k} \in W\} \cup (0,0)$,
Σ var-cov matrix W set of portfolios s.t. $e^T w = 1\ \forall w \in W$ and assuming that $\exists \hat{w} \in W \mid \mu^T \hat{w} > 0$

Proposed formulation

$$\sum_{i=1}^{n} \sum_{j=i}^{n} \Sigma_{ij} \left(\sum_{k=0}^{p-1} c_k x_{ik}\right)\left(\sum_{k=0}^{p-1} c_k x_{jk}\right)$$

With:

$x_i \in \{0,1\},\ c_k = \frac{2^k}{10}$ allocation discretization coefficient, $c_{p-1} = \frac{1}{\mu_{\min}} - \sum_{l=0}^{p-2} c_k$, discretization step $=$ 0.1 and assuming $\mu_i > 0\ \forall i$

STEP 2: DIVERSIFICATION

Need definition

The need is to diversify the portfolio selection over multiple sectors

Mathematical model

$$\sum_{i=1}^{n}\sum_{k=0}^{p-1} f_i x_{ik} + \sum_{i=1}^{n}\sum_{k=0}^{p-1}\sum_{j=1}^{n}\sum_{l=0}^{p-1} D_{ij} x_{ik} x_{jl}$$

Where:

$f\_i < 0 \ \forall i = 1, \dots, n$ vector of components that drive the solution to invest a positive quantity and $D \in \mathbb{B}^{n \times n}$ $s.t.$

$$D_{ij} = \begin{cases} 1 & \text{asset i and j belong to the same sector} \\ 0 & \text{otherwise} \end{cases}$$

In general, this objective function is not convex!

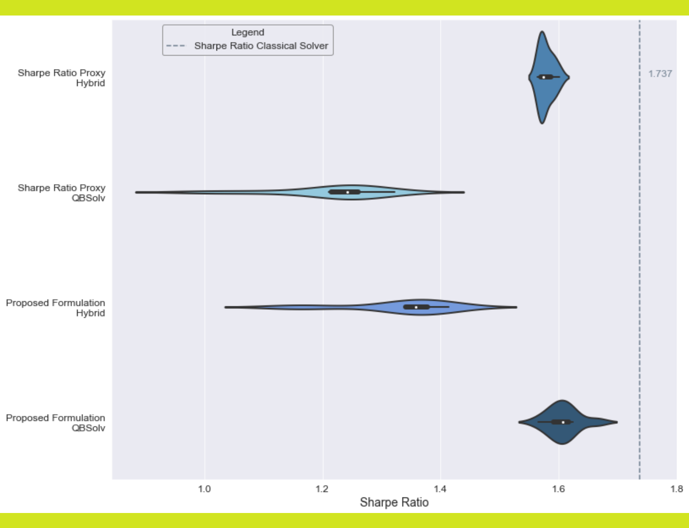
OVERALL PROPOSED QUBO MODEL

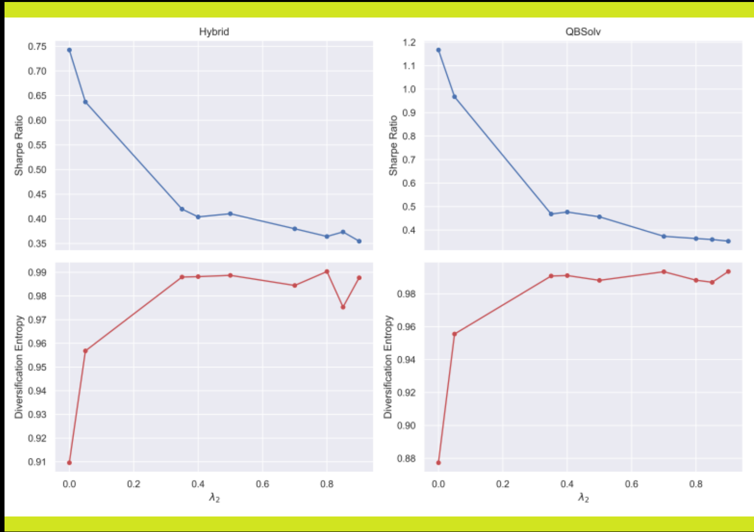
$$QUBO = \lambda_0 H_0(x) + \lambda_1 H_1(x) + \lambda_2 H_2(x)$$

H_0(x) **Sharpe Ratio Component**	$\sum_{i=1}^{n}\sum_{j=i}^{n}\Sigma_{ij}\left(\sum_{k=0}^{p-1} c_k x_{ik}\right)\left(\sum_{k=0}^{p-1} c_k x_{jk}\right)$
H_1(x) **Constraint Component**	$\left(\sum_{i=1}^{n}\sum_{k=0}^{p-1} \mu_i c_k x_{ik} - 1\right)^2$
H_2(x) **Diversification Component**	$\sum_{i=1}^{n}\sum_{k=0}^{p-1} f_i x_{ik} + \sum_{i=1}^{n}\sum_{k=0}^{p-1}\sum_{j=1}^{n}\sum_{l=0}^{p-1} D_{ij} x_{ik} x_{jl}$

RESULTS AND CONCLUSIONS

RESULTS





Best results among the QUBO are obtained on the Proposed Formulation for Sharpe Ratio Maximization

QUBO formulation provides flexible framework for trade-off between Sharpe Ratio and diversification

QUBO AND HPC INFRASTRUCTURES

- QUBO is a model that can greatly benefit from HPC infrastructure
- Complex nonlinear problems that fit the QUBO formulation can benefit from massive parallelization techniques, by:
  - Quickly building large QUBO matrices
  - Performing parallel computations on the matrix
  - Avoiding connectivity limitations of current Quantum Computers
- MegaQUBO is Data Reply's accelerator to handle large-scale QUBO problems in a reduced amount of time, leveraging HPC infrastructures

WRAP-UP AND CONCLUSIONS

- We introduced the Portfolio Optimization problem in a common, yet classically challenging, setting
- We presented a novel QUBO formulation to solve the problem and discussed the results in regards to hybrid Quantum-Classical and High-Performance computing
- Future possible extentions of this work include:
  - Handle the Portfolio Optimization problem with MegaQUBO
  - Deeper investigation of alternative algorithms in comparison to solving our QUBO formulation

THANK YOU FOR THE ATTENTION

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