DIVERSIFYING INVESTMENTS AND MAXIMIZING SHARPE RATIO: A NOVEL QUBO FORMULATION

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AGENDA

- Portfolio optimization: problem overview and data description
- Novel formulation: QUBO for maximization of diversification and Sharpe Ratio
- Results and conclusions
PORTFOLIO OPTIMIZATION: PROBLEM OVERVIEW
PORTFOLIO OPTIMIZATION

Overview & Goals

- Identify the optimal investments on a basket of assets satisfying desired conditions on the expected return and portfolio risk
- Provide a diversified portfolio, allocating budget on multiple sectors

First component of objective function: Sharpe Ratio maximization

\[
\text{Sharpe Ratio} = \frac{w^T \mu}{\sqrt{w^T \Sigma w}}
\]

- \( \Sigma \) variance-covariance matrix
- \( w_i \in [0,1] \) amount invested
- on \( i-th \) asset

Second component: diversification maximization which makes the problem, in general, non convex, and thus hard to solve with standard techniques
DATA DESCRIPTION

- Historical S&P500 asset prices taken as daily adjusted closes from 2010 to 2020

- Returns and covariance matrix are annualized, i.e. their value is multiplied by 252 (assumed number of working days in a year)

- Historical returns are considered in the form of log-returns instead of simple returns, as suggested by Shapiro – Q-Q plot shown in figure as a qualitative comparison of the distributions
NOVEL FORMULATION: QUBO FOR MAXIMIZATION OF DIVERSIFICATION AND SHARPE RATIO
QUBO MODEL

- Quantum annealers minimize Ising Energy
  
  \[ H = \sum_i h_i S_i + \sum_{ij} J_{ij} S_i S_j \]

- Changing variables, the problem is equivalently written as a QUBO:
  
  \[ \min E(x|Q) = \sum_{i\leq j} x_i Q_{ij} x_j \]

**Q** | Quadratic: the highest power of variables is \( x^2 \)

**U** | Unconstrained: no external constraints are applied

**B** | Binary: variables are binary \( \{0,1\} \)

**O** | Optimization: Minimization of an objective function (the energy)
**STEP 1: REFORMULATION**

**Portfolio selection**
(current model from literature)

\[
\sum_{i=1}^{n} a_i q_i + \sum_{i=1}^{n} \sum_{j=i+1}^{n} b_{ij} q_i q_j
\]

With:

\( q_i \in \{0,1\}, \ a_i = \frac{\mu_i}{\sigma_i}, \ b_{ij} = \rho_{ij}, \) where \( \rho_{ij} \) is the correlation between asset \( i \) and asset \( j \)

**Proxy formulation**

\[
\sum_{i=1}^{n} a_i \left( \sum_{l=0}^{p-1} d_l x_{il} \right) + \sum_{i=1}^{n} \sum_{j=i+1}^{n} b_{ij} \left( \sum_{l=0}^{p-1} d_l x_{il} \right) \left( \sum_{l=0}^{p-1} d_l x_{jl} \right)
\]

With:

\( x_i \in \{0,1\}, \ a_i = \frac{\mu_i}{\sigma_i}, \ b_{ij} = \rho_{ij}, \ d_l = \frac{z_l}{500} \) allocation discretization coefficient and \( d_{p-1} = 1 - \sum_{l=0}^{p-2} d_l \), investment granularity = 0.2%
**STEP 1: REFORMULATION**

**Portfolio selection**
(current model from literature)

\[
\sum_{i} a_{i} q_{i} + \sum_{i} \sum_{j} b_{ij} q_{i} q_{j}
\]

With:
- \( q_{i} \in \{0,1\} \), \( a_{i} = \frac{\mu_{i}}{\sigma_{i}} \) and \( b_{ij} = \rho_{ij} \), where \( \rho_{ij} \) is the correlation between asset \( i \) and asset \( j \)

**Proxy formulation**

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\sum_{i=1}^{n} a_{i} \left( \sum_{l=0}^{p-1} d_{l} x_{il} \right) + \sum_{i=1}^{n} \sum_{j=i+1}^{n} b_{ij} \left( \sum_{l=0}^{p-1} d_{l} x_{il} \right) \left( \sum_{l=0}^{p-1} d_{l} x_{jl} \right)
\]

With:
- \( x_{i} \in \{0,1\} \), \( a_{i} = \frac{\mu_{i}}{\sigma_{i}} \), \( b_{ij} = \rho_{ij} \), \( d_{l} = \frac{2^{l}}{500} \) allocation discretization coefficient and \( d_{p-1} = 1 - \sum_{l=0}^{p-2} d_{l} \), investment granularity = 0.2%

*Proxy Formulation does not mathematically equate to the maximization of the Sharpe Ratio as it is defined in the classical problem!*
### Portfolio selection

**Classical model (quadratic formulation)**

\[
\min y^T \Sigma y \\
s.t. \ y^T y = 1, \quad (y, k) \in Y
\]

With:
- \( Y := \{ y \in R^N, k \in R \mid k > 0, w = \frac{y}{k} \in W \} \cup (0, 0) \),
- \( \Sigma \) var-cov matrix \( W \) set of portfolios s.t. \( e^T w = 1 \forall w \in W \) and assuming that \( \exists \tilde{w} \in W \mid \mu^T \tilde{w} > 0 \)

**Proposed formulation**

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=0}^{p-1} \Sigma_{ij} \left( \sum_{k=0}^{p-1} c_{ik} x_{ik} \right) \left( \sum_{k=0}^{p-1} c_{jk} x_{jk} \right)
\]

With:
- \( x_i \in \{0,1\}, c_k = \frac{2^k}{10} \) allocation discretization coefficient, \( c_{p-1} = \frac{1}{\mu_{\min}} - \sum_{l=0}^{p-2} c_k \),
- discretization step = 0.1 and assuming \( \mu_i > 0 \forall i \)

### Proxy formulation

\[
\sum_{i=1}^{n} a_i \left( \sum_{l=0}^{p-1} d_i x_{il} \right) + \sum_{i=1}^{n} \sum_{j=l+1}^{n} b_{ij} \left( \sum_{l=0}^{p-1} d_i x_{il} \right) \left( \sum_{l=0}^{p-1} d_j x_{jl} \right)
\]

With:
- \( x_i \in \{0,1\}, a_i = \frac{\mu_i}{\sigma_i}, b_{ij} = \rho_{ij}, d_{i} = \frac{x_i}{500} \) allocation discretization coefficient and \( d_{p-1} = 1 - \sum_{l=0}^{p-1} d_i \), investment granularity = 0.2%

**STEP 1: REFORMULATION**
STEP 2: DIVERSIFICATION

**Need definition**

The need is to diversify the portfolio selection over multiple sectors.

**Mathematical model**

\[
\sum_{i=1}^{n} \sum_{k=0}^{p-1} \tilde{f}_i x_{ik} + \sum_{i=1}^{n} \sum_{k=0}^{p-1} \sum_{j=1}^{n} \sum_{l=0}^{p-1} D_{ij} x_{ik} x_{jl}
\]

Where:

- \( f_i < 0 \) for all \( i = 1, ..., n \) vector of components that drive the solution to invest a positive quantity and \( D \in \mathbb{R}^{n \times n} \).
- \( D_{ij} = \begin{cases} 1 & \text{asset } i \text{ and } j \text{ belong to the same sector} \\ 0 & \text{otherwise} \end{cases} \)

In general, this objective function is not convex!
# OVERALL PROPOSED QUBO MODEL

\[ QUBO = \lambda_0 H_0(x) + \lambda_1 H_1(x) + \lambda_2 H_2(x) \]

### $H_0(x)$ Sharpe Ratio Component

\[
\sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{k=0}^{p-1} c_k x_{ik} \left( \sum_{k=0}^{p-1} c_k x_{jk} \right)
\]

### $H_1(x)$ Constraint Component

\[
\left( \sum_{i=1}^{n} \sum_{k=0}^{p-1} \mu_i c_k x_{ik} - 1 \right)^2
\]

### $H_2(x)$ Diversification Component

\[
\sum_{i=1}^{n} \sum_{k=0}^{p-1} f_{ik} x_{ik} + \sum_{i=1}^{n} \sum_{k=0}^{p-1} \sum_{j=1}^{n} \sum_{l=0}^{p-1} D_{ij} x_{ik} x_{jl}
\]
RESULTS AND CONCLUSIONS
RESULTS

Best results among the QUBO are obtained on the Proposed Formulation for Sharpe Ratio Maximization

QUBO formulation provides flexible framework for trade-off between Sharpe Ratio and diversification
QUBO AND HPC INFRASTRUCTURES

- QUBO is a model that can greatly benefit from HPC infrastructure

- Complex nonlinear problems that fit the QUBO formulation can benefit from massive parallelization techniques, by:
  - Quickly building large QUBO matrices
  - Performing parallel computations on the matrix
  - Avoiding connectivity limitations of current Quantum Computers

- MegaQUBO is Data Reply’s accelerator to handle large-scale QUBO problems in a reduced amount of time, leveraging HPC infrastructures
WRAP-UP AND CONCLUSIONS

- We introduced the Portfolio Optimization problem in a common, yet classically challenging, setting.

- We presented a novel QUBO formulation to solve the problem and discussed the results in regards to hybrid Quantum-Classical and High-Performance computing.

- Future possible extensions of this work include:
  - Handle the Portfolio Optimization problem with MegaQUBO
  - Deeper investigation of alternative algorithms in comparison to solving our QUBO formulation.
THANK YOU FOR THE ATTENTION