# INTRODUCTION TO QUANTUM ANNEALING

Formulating and solve QUBO Problems

*Daniele Ottaviani*

CINECA QUANTUM COMPUTING LAB

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- **Quantum Annealing is a quantum algorithm capable of solving optimization problems**



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PHYSICAL REVIEW E VOLUME 58, NUMBER 5 NOVEMBER 1998

## Quantum annealing in the transverse Ising model

Tadashi Kadowaki and Hidetoshi Nishimori
*Department of Physics, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo 152-8551, Japan*
(Received 30 April 1998)

We introduce quantum fluctuations into the simulated annealing process of optimization problems, aiming at faster convergence to the optimal state. Quantum fluctuations cause transitions between states and thus play the same role as thermal fluctuations in the conventional approach. The idea is tested by the transverse Ising model, in which the transverse field is a function of time similar to the temperature in the conventional method. The goal is to find the ground state of the diagonal part of the Hamiltonian with high accuracy as quickly as possible. We have solved the time-dependent Schrödinger equation numerically for small size systems with various exchange interactions. Comparison with the results of the corresponding classical (thermal) method reveals that the quantum annealing leads to the ground state with much larger probability in almost all cases if we use the same annealing schedule. [S1063-651X(98)02910-9]

PACS number(s): 05.30.−d, 75.10.Nr, 89.70.+c

### I. INTRODUCTION

The technique of simulated annealing (SA) was first proposed by Kirkpatrick *et al.* [1] as a general method to solve optimization problems. The idea is to use thermal fluctuations to allow the system to escape from local minima of the cost function so that the system reaches the global minimum under an appropriate annealing schedule (the rate of decrease of temperature). If the temperature is decreased too quickly, the system may become trapped in a local minimum. Too slow annealing, on the other hand, is practically useless although such a process would certainly bring the system to the global minimum. Geman and Geman proved a theorem on the annealing schedule for a generic problem of combinatorial optimization [2]. They showed that any system reaches the global minimum of the cost function asymptotically if the temperature is decreased as $T=c/\ln t$ or slower, where $c$ is a constant determined by the system size and other structures of the cost function. This bound on the annealing schedule may be the optimal one under generic con-

specific model system, rather than to develop a general argument, to gain insight into the role of quantum fluctuations in the situation of optimization problem. Quantum effects have been found to play a very similar role to thermal fluctuations in the Hopfield model in a transverse field in thermal equilibrium [5]. This observation motivates us to investigate dynamical properties of the Ising model under quantum fluctuations in the form of a transverse field. We therefore discuss in this paper the transverse Ising model with a variety of exchange interactions. The transverse field controls the rate of transition between states and thus plays the same role as the temperature does in SA. We assume that the system has no thermal fluctuations in the QA context and the term ‘‘ground state’’ refers to the lowest-energy state of the Hamiltonian without the transverse field term.

Static properties of the transverse Ising model have been investigated quite extensively for many years [6]. There have, however, been very few studies on the dynamical behavior of the Ising model with a transverse field. We refer to the work by Sato *et al.* who carried out quantum Monte

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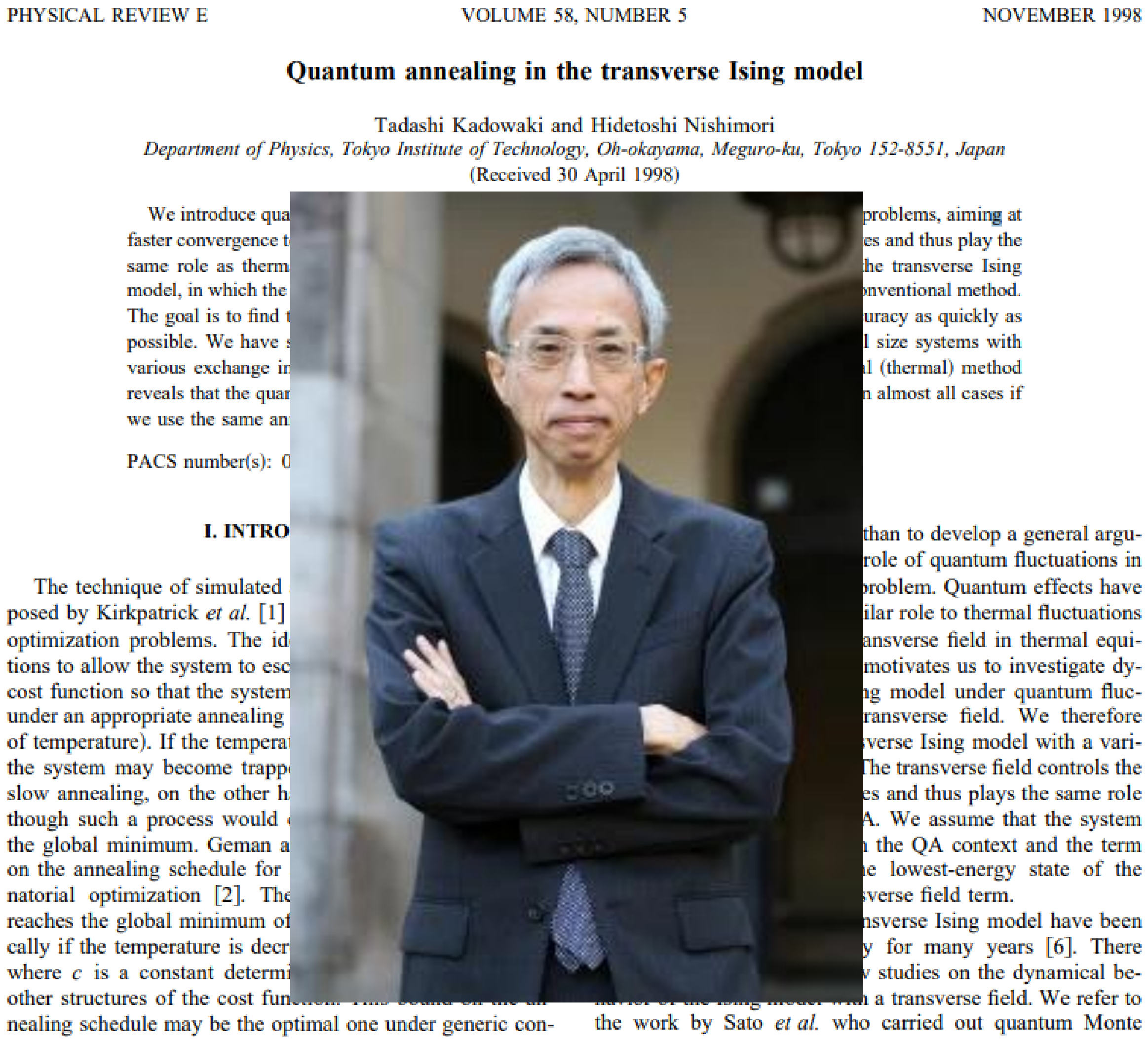
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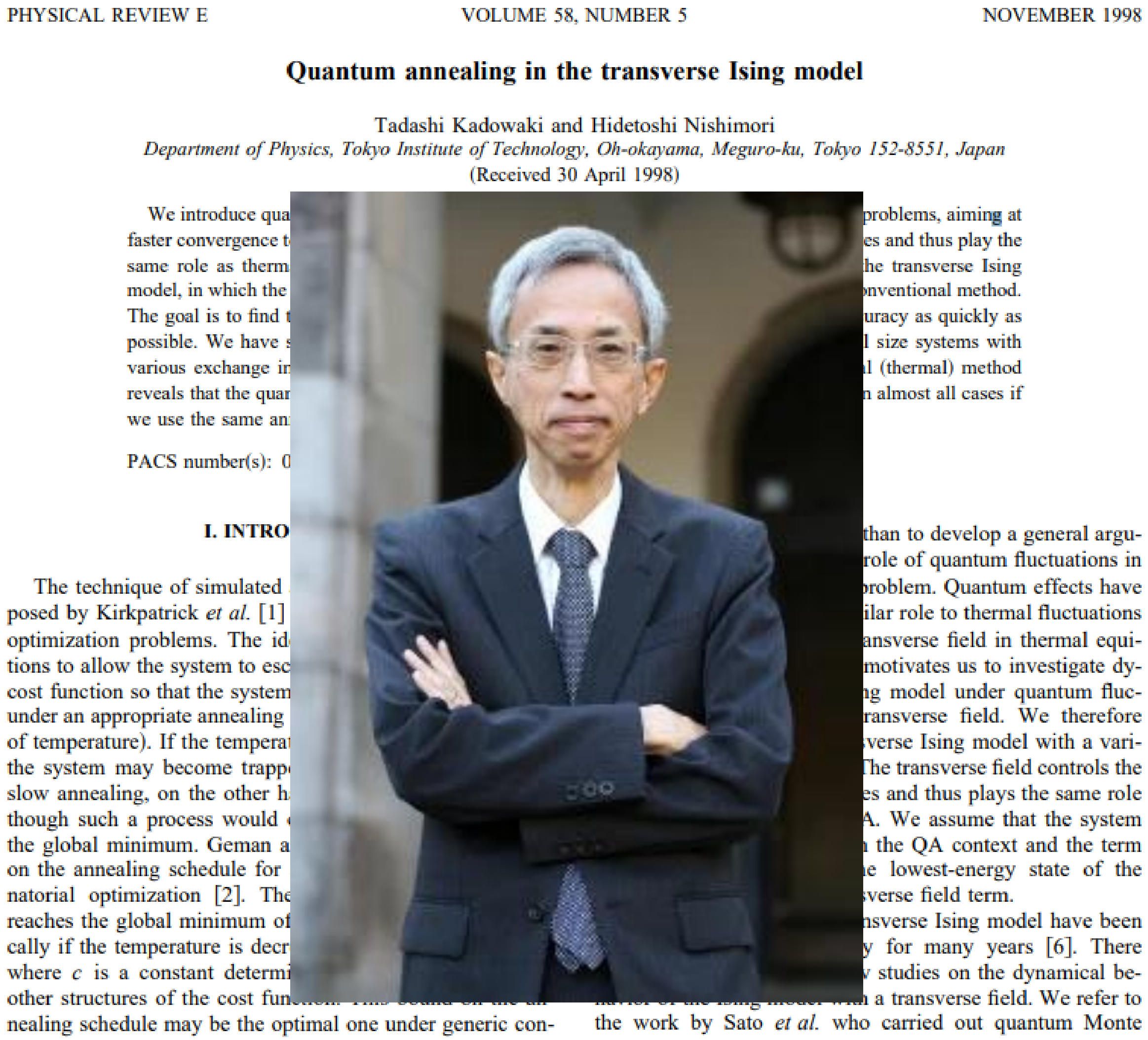

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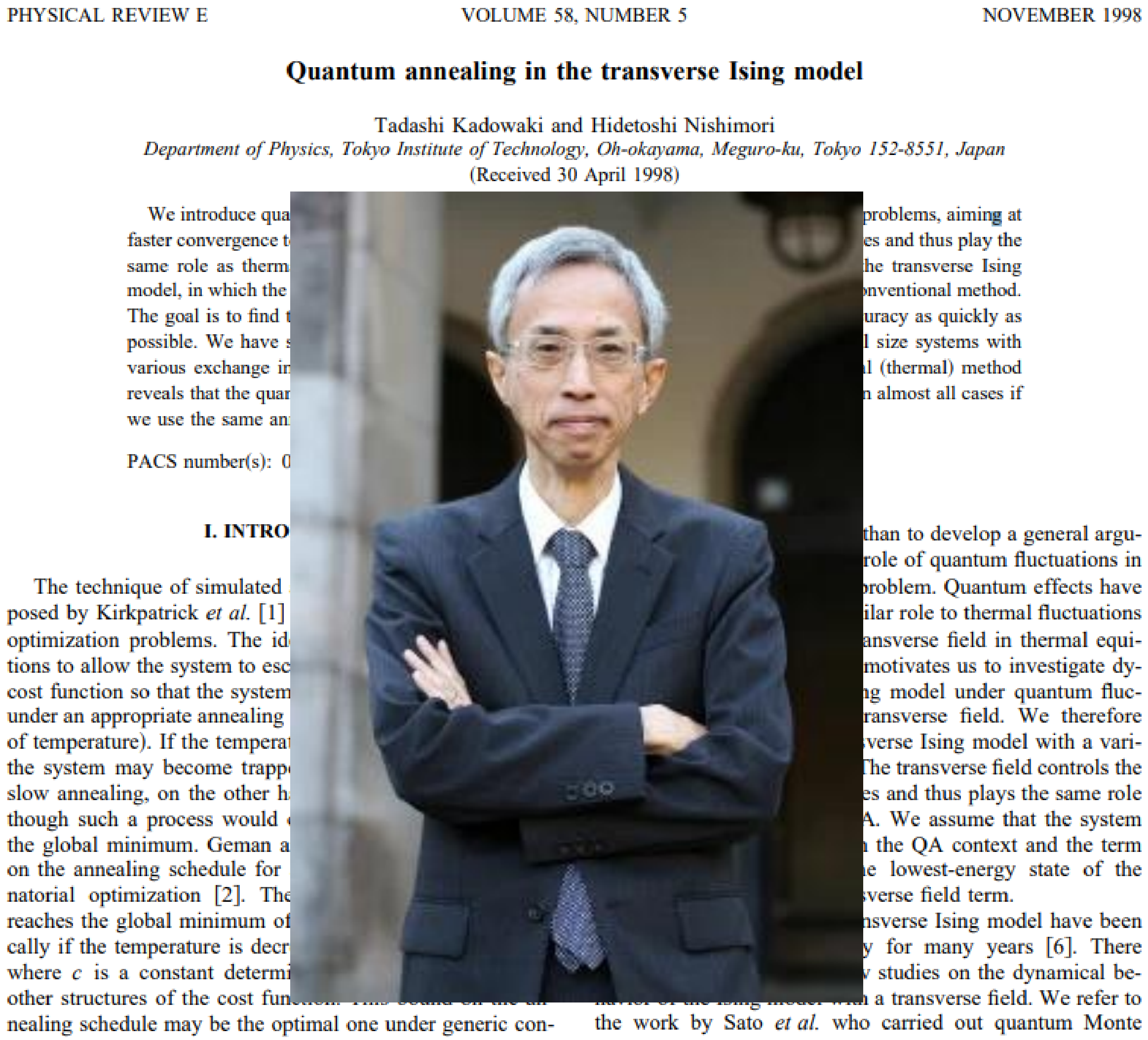

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# Annealing Algorithms

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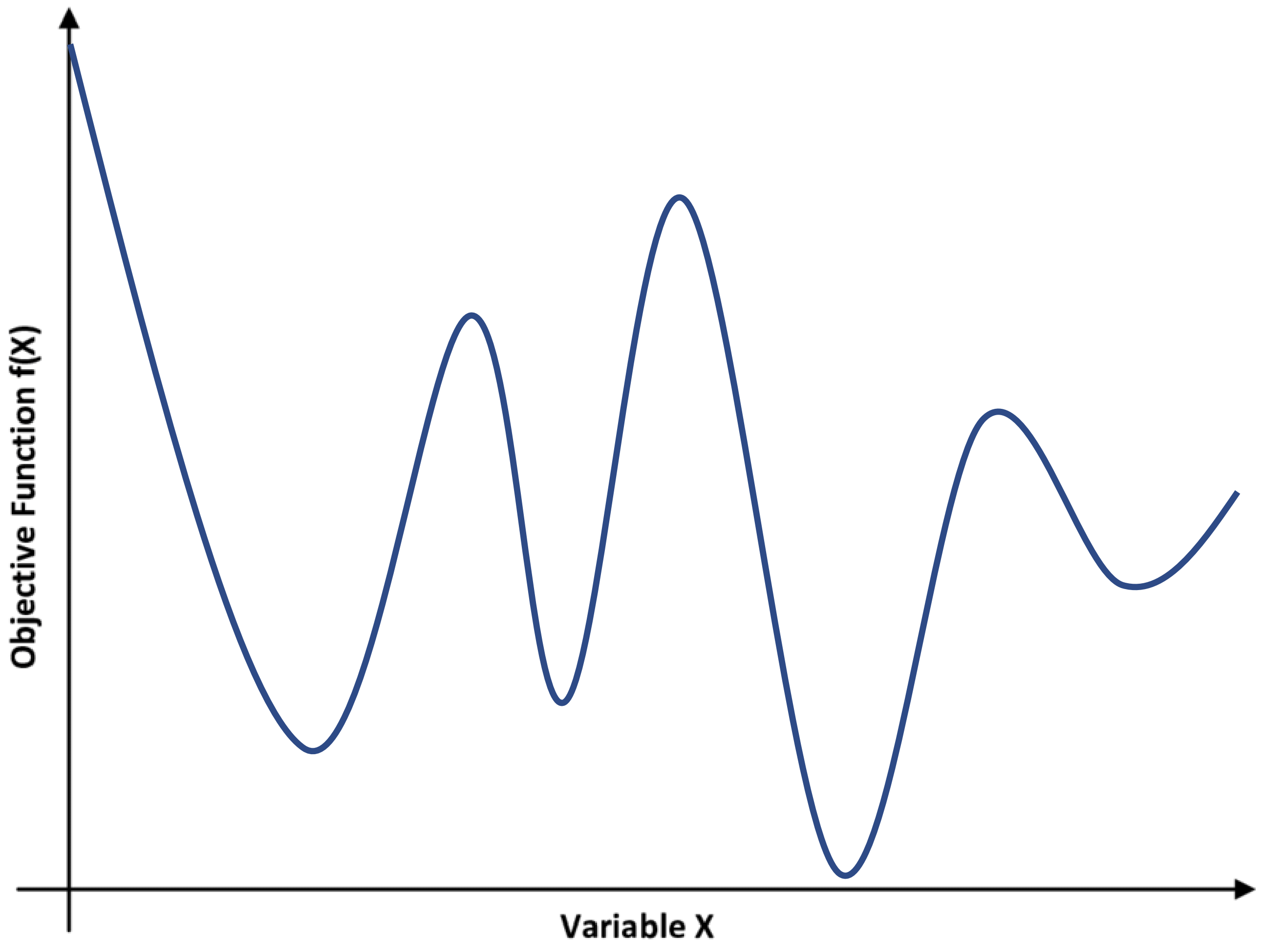

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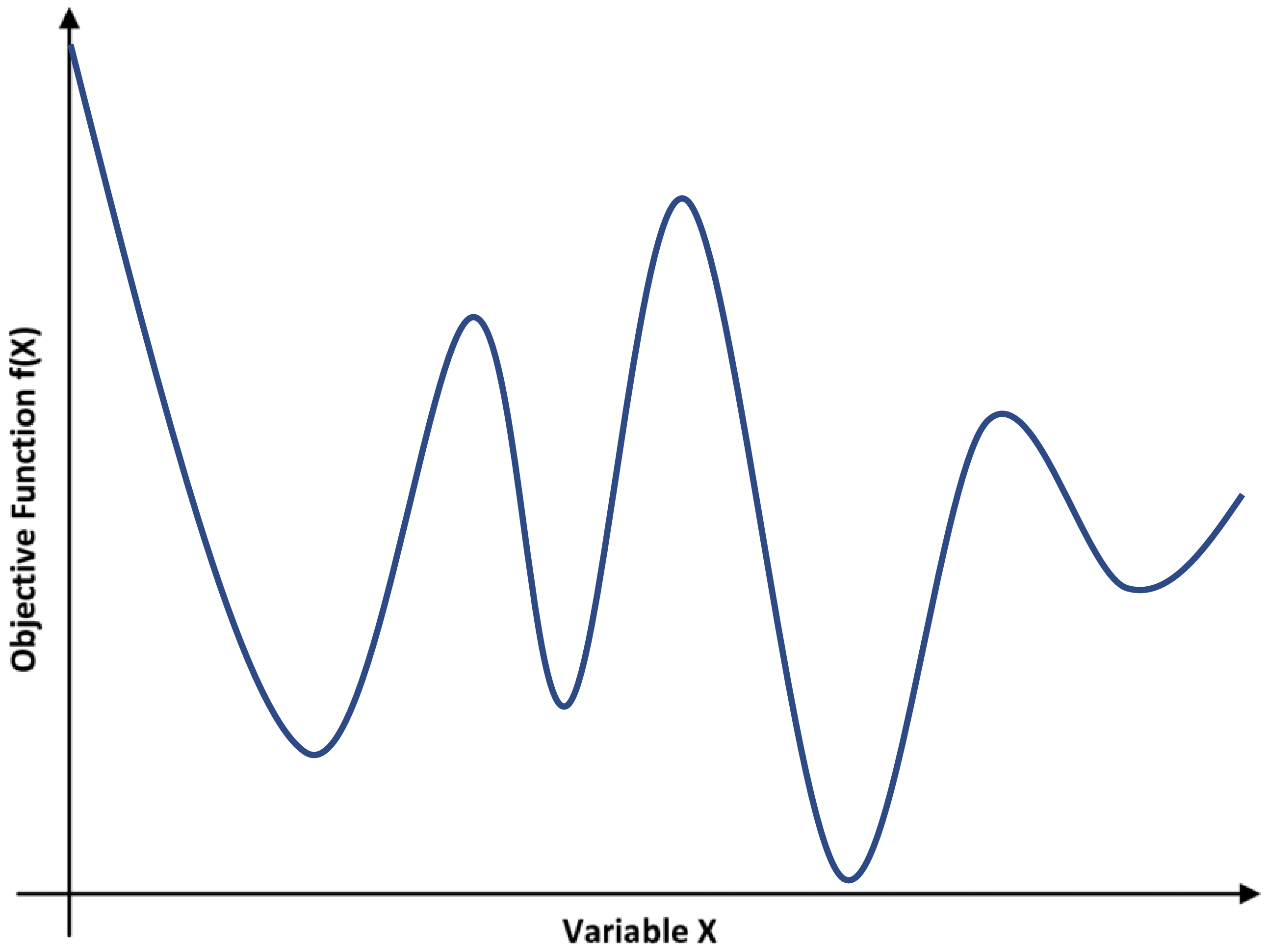

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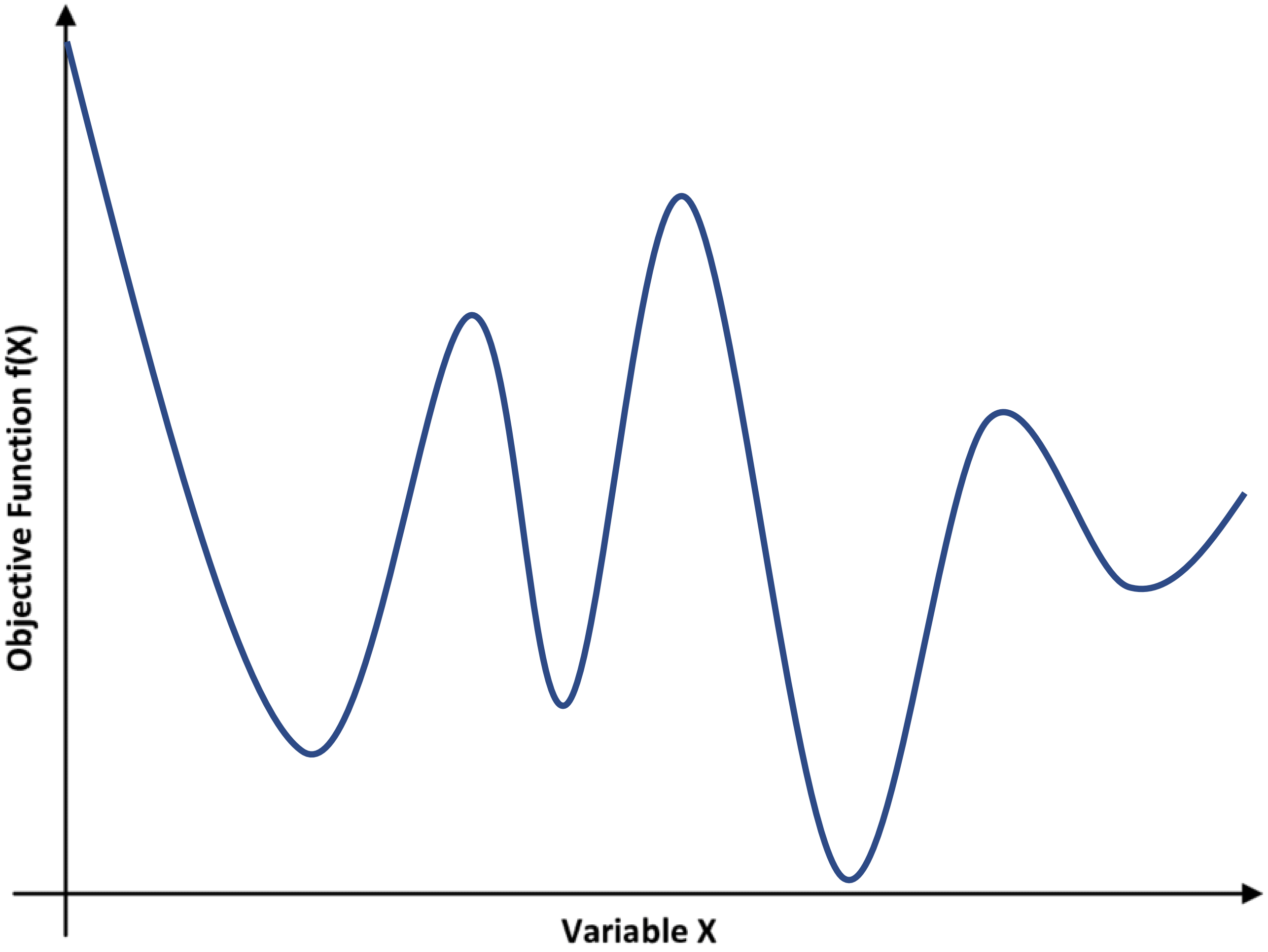

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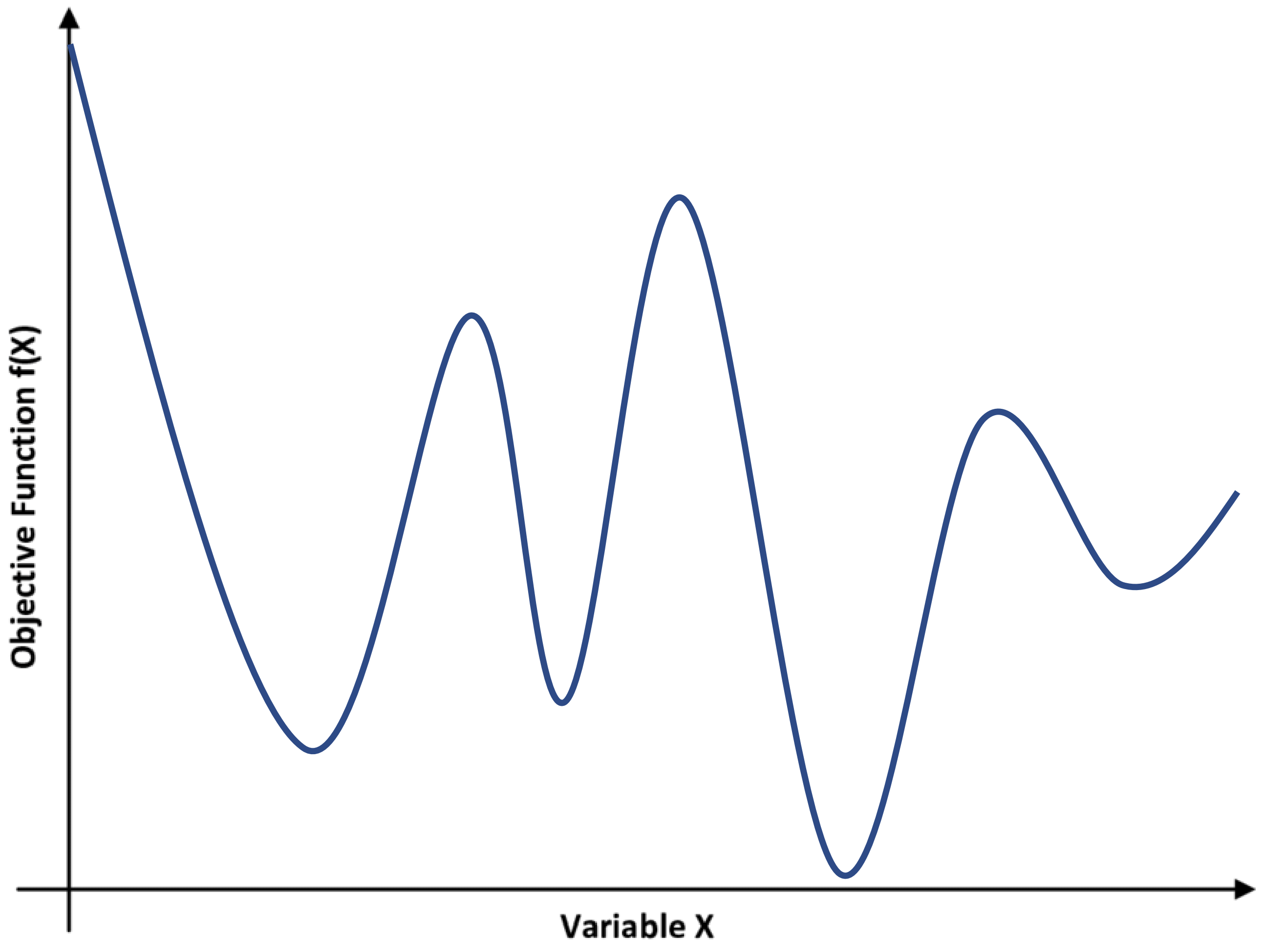

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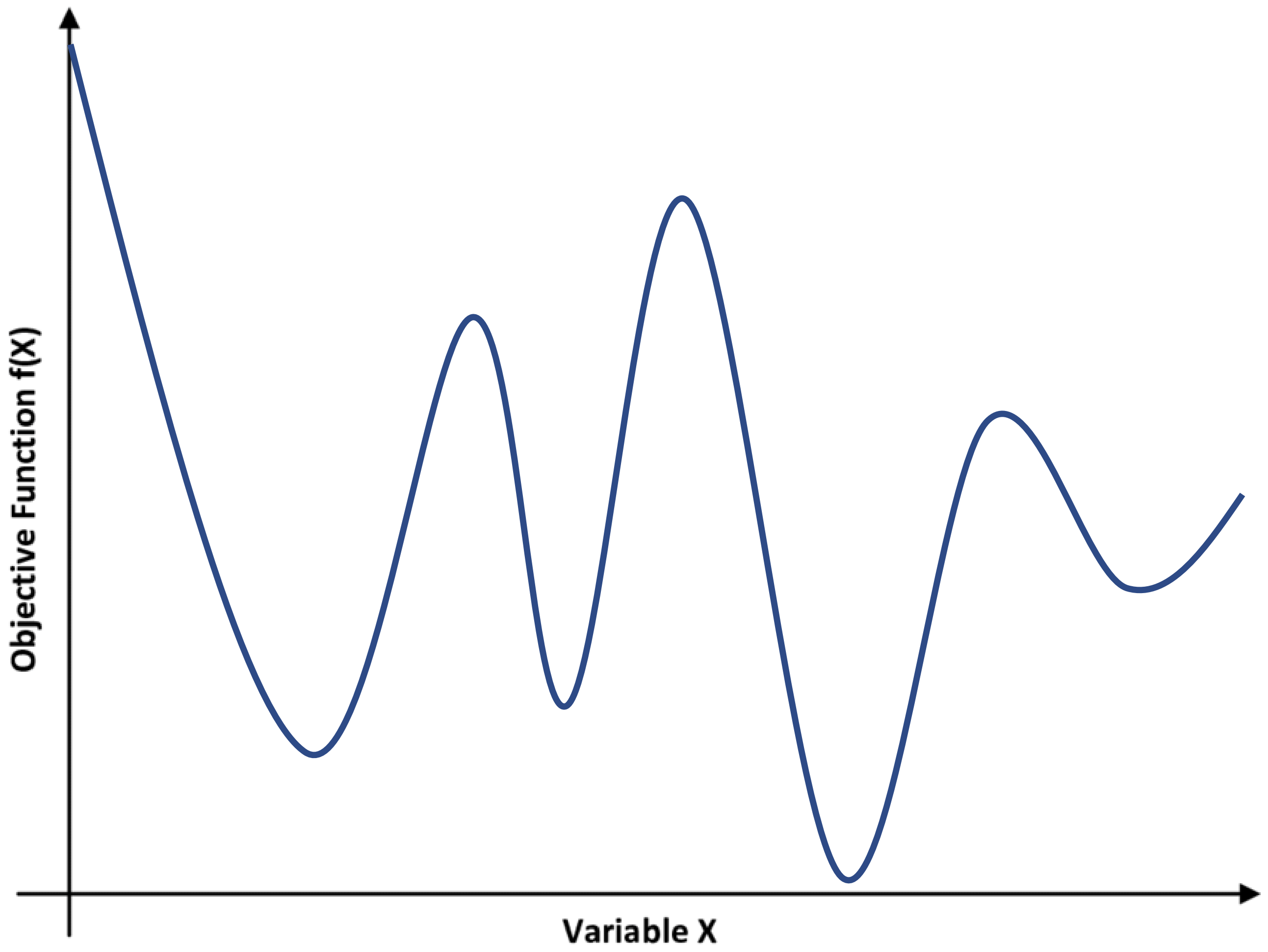

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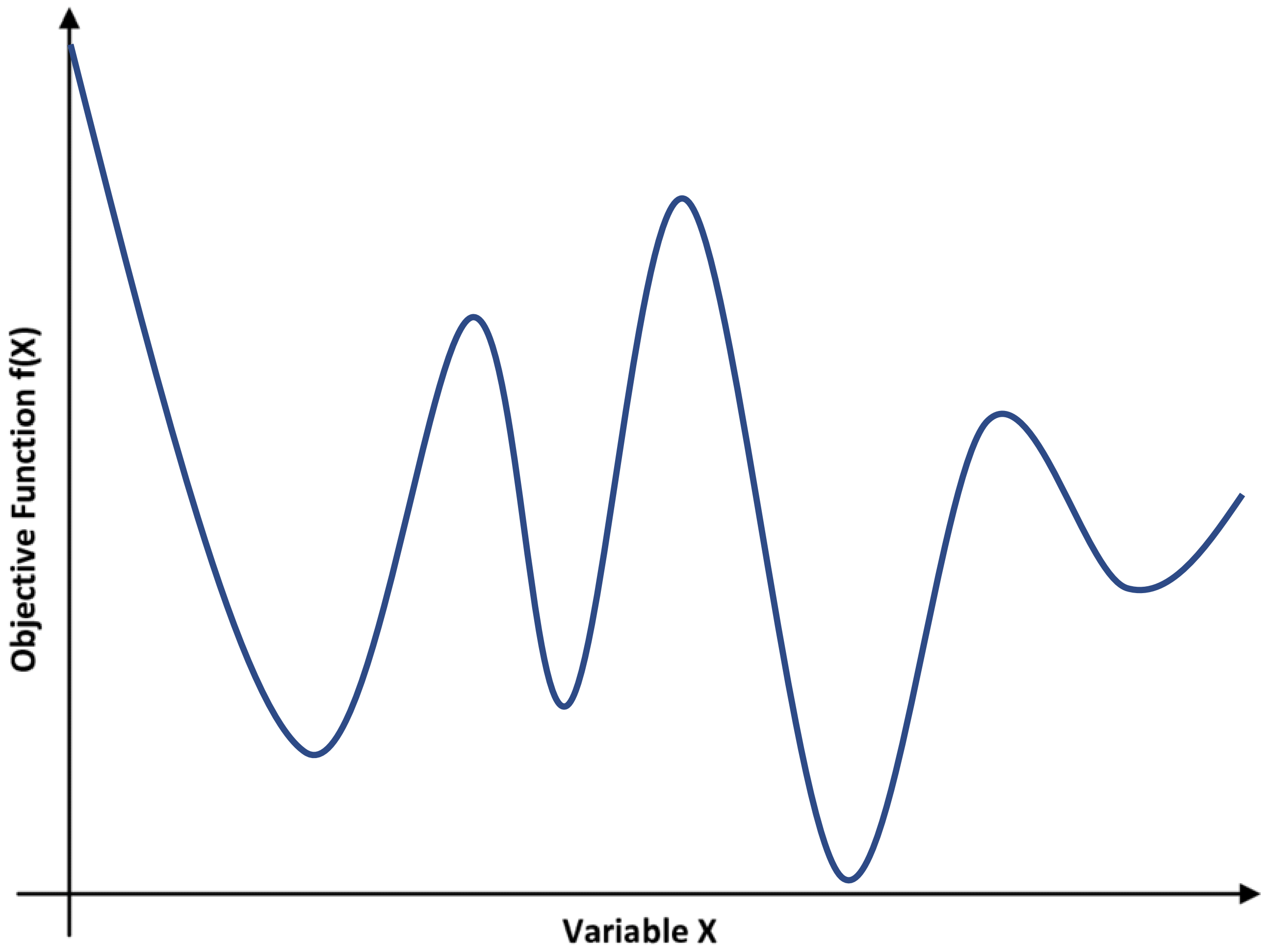

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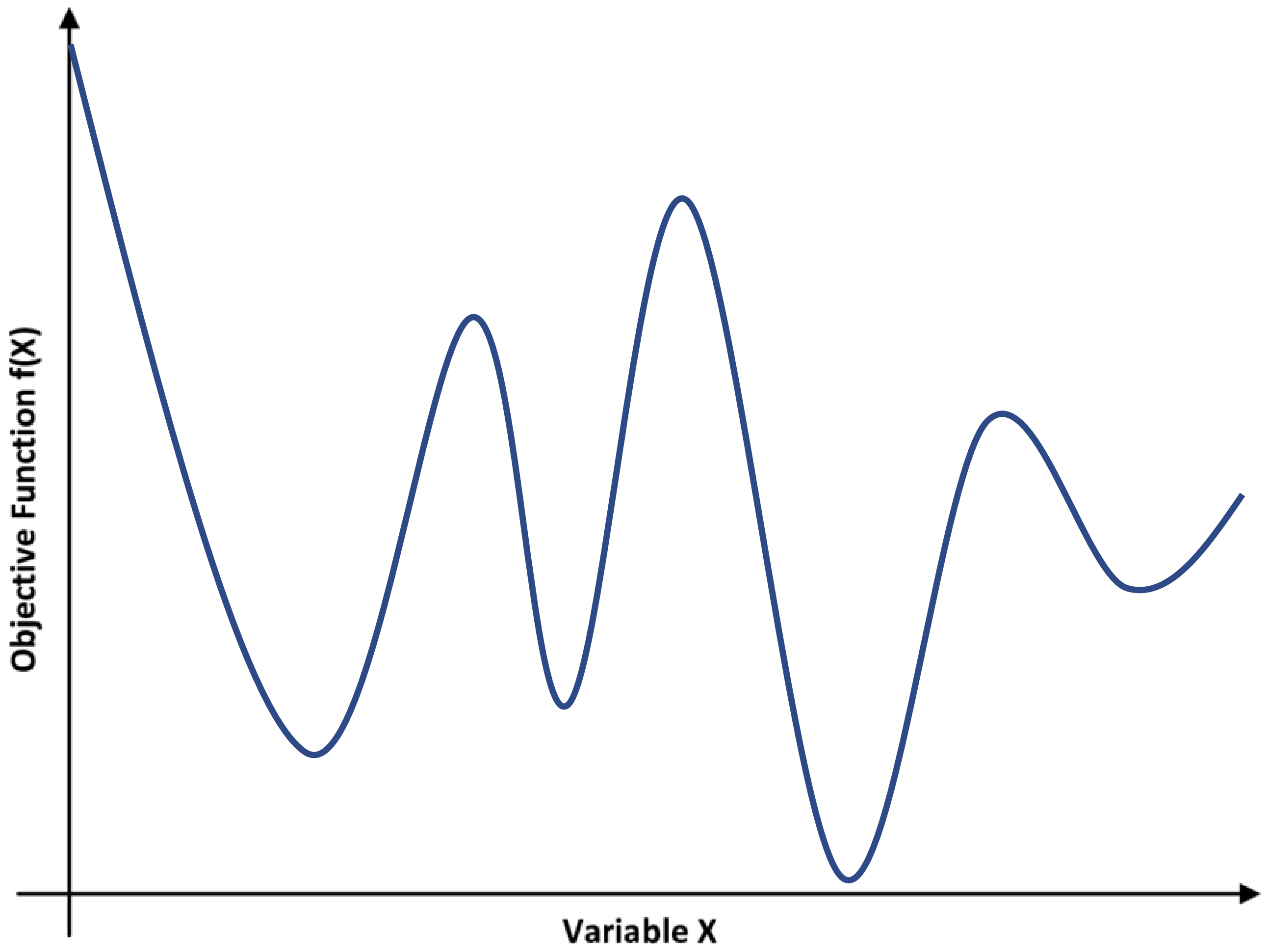

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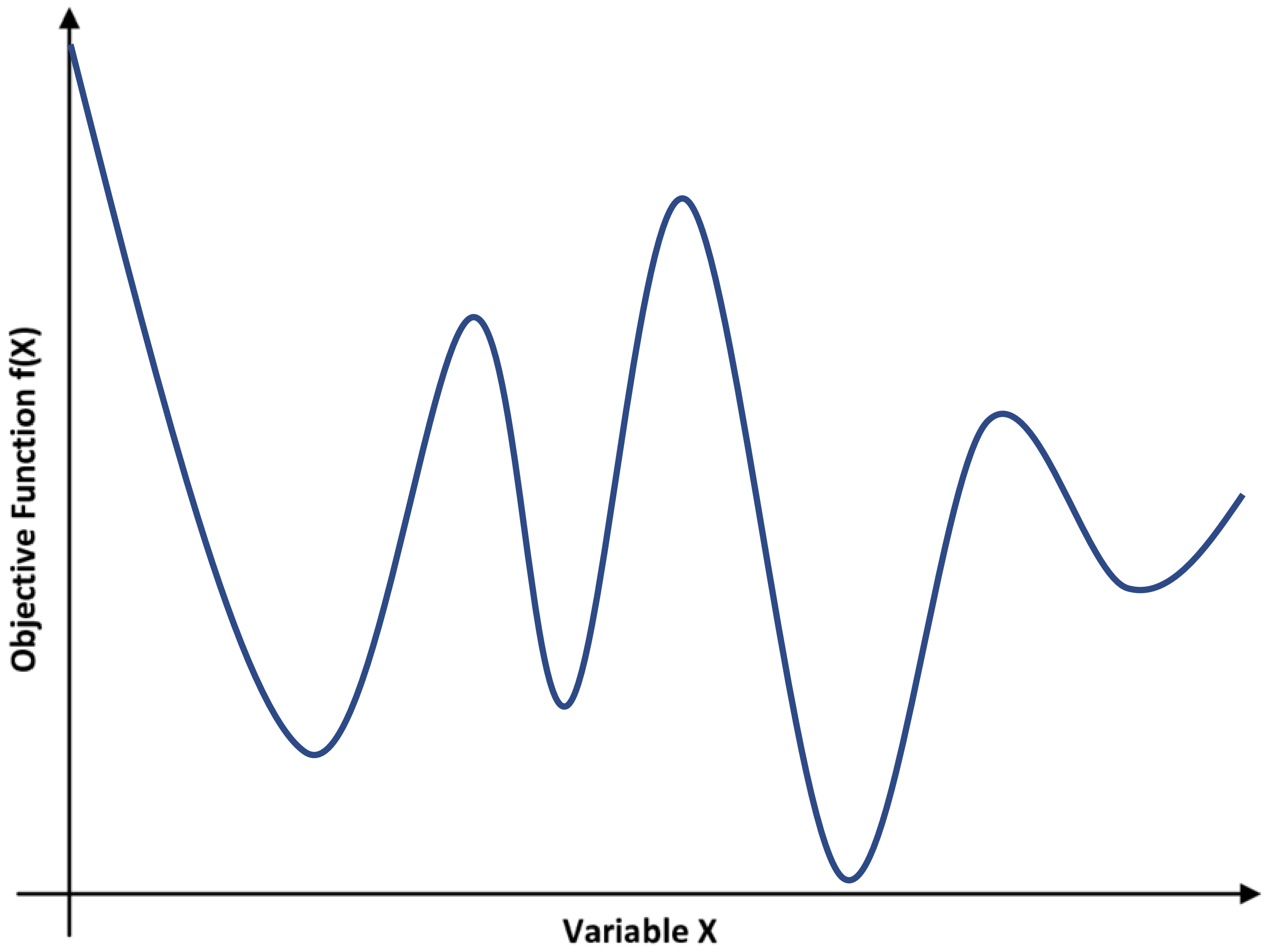

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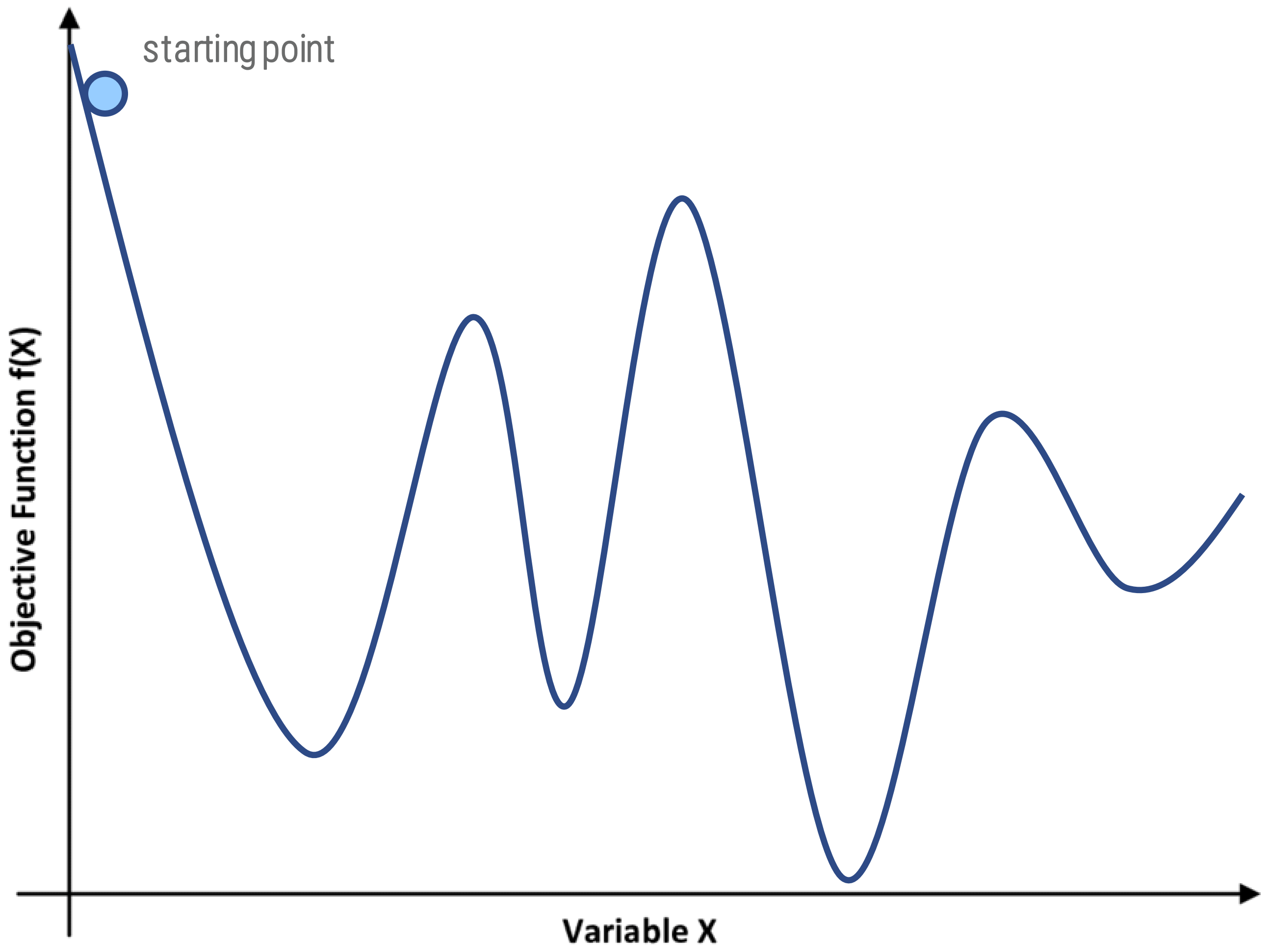

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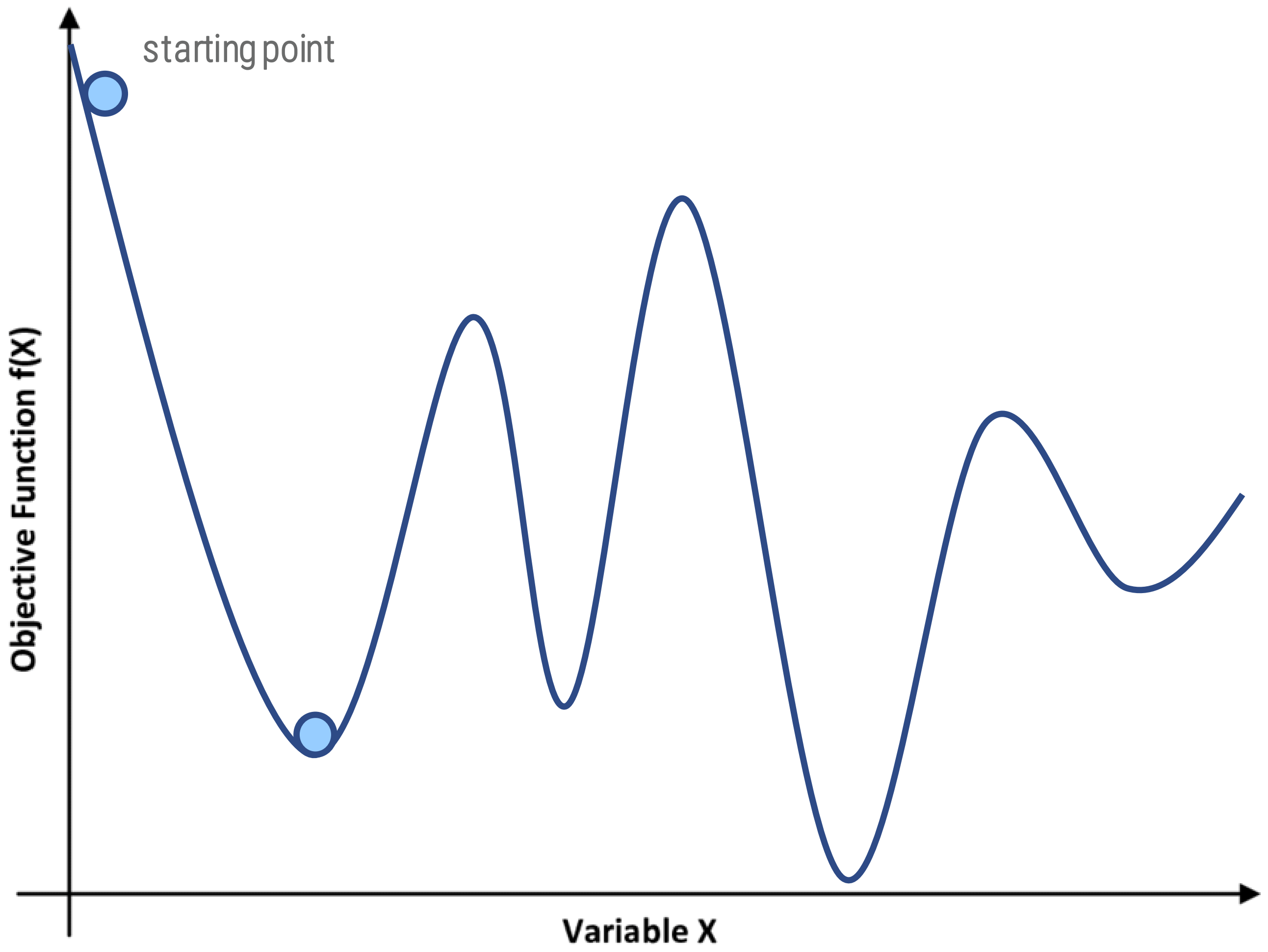

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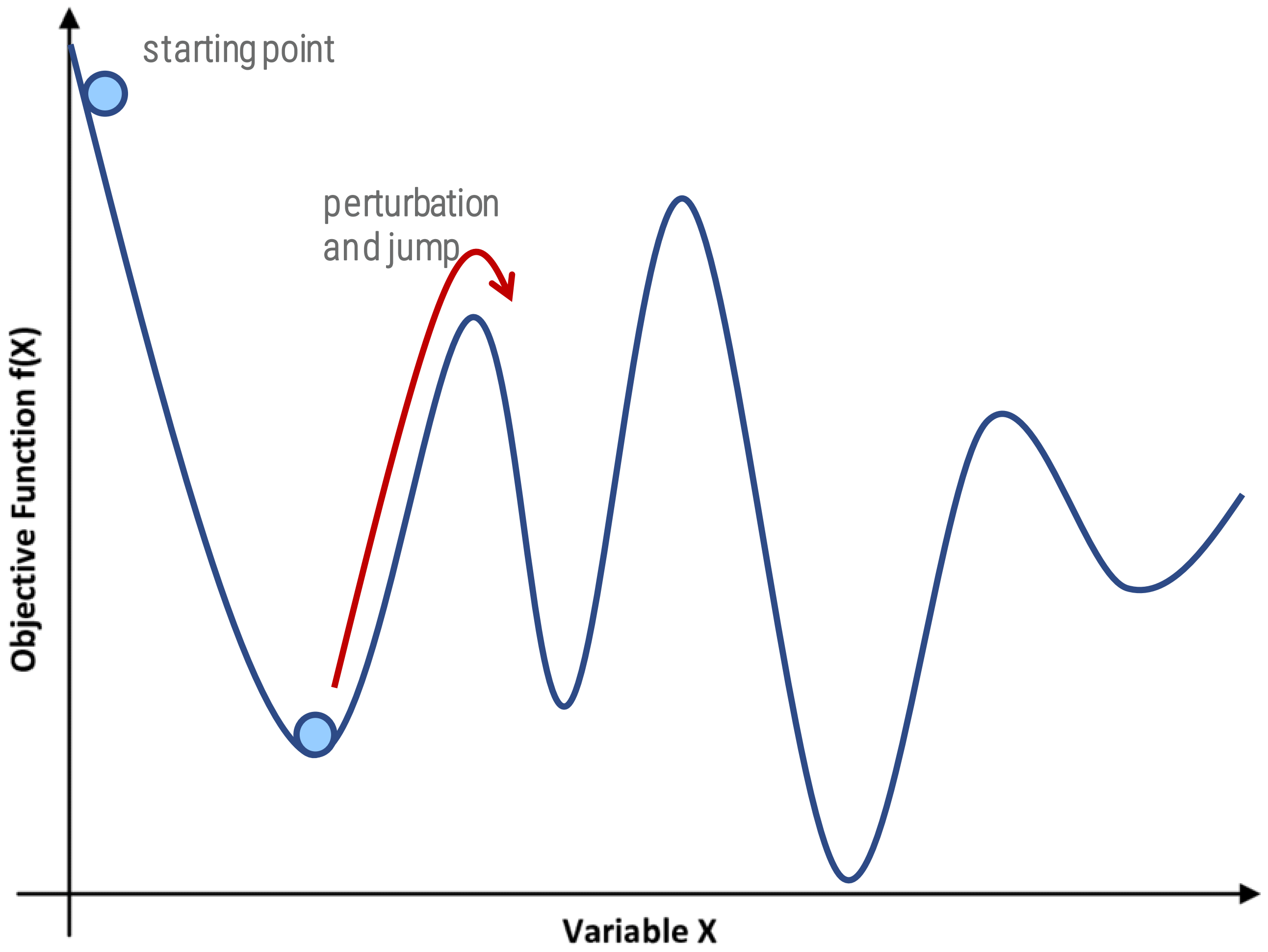


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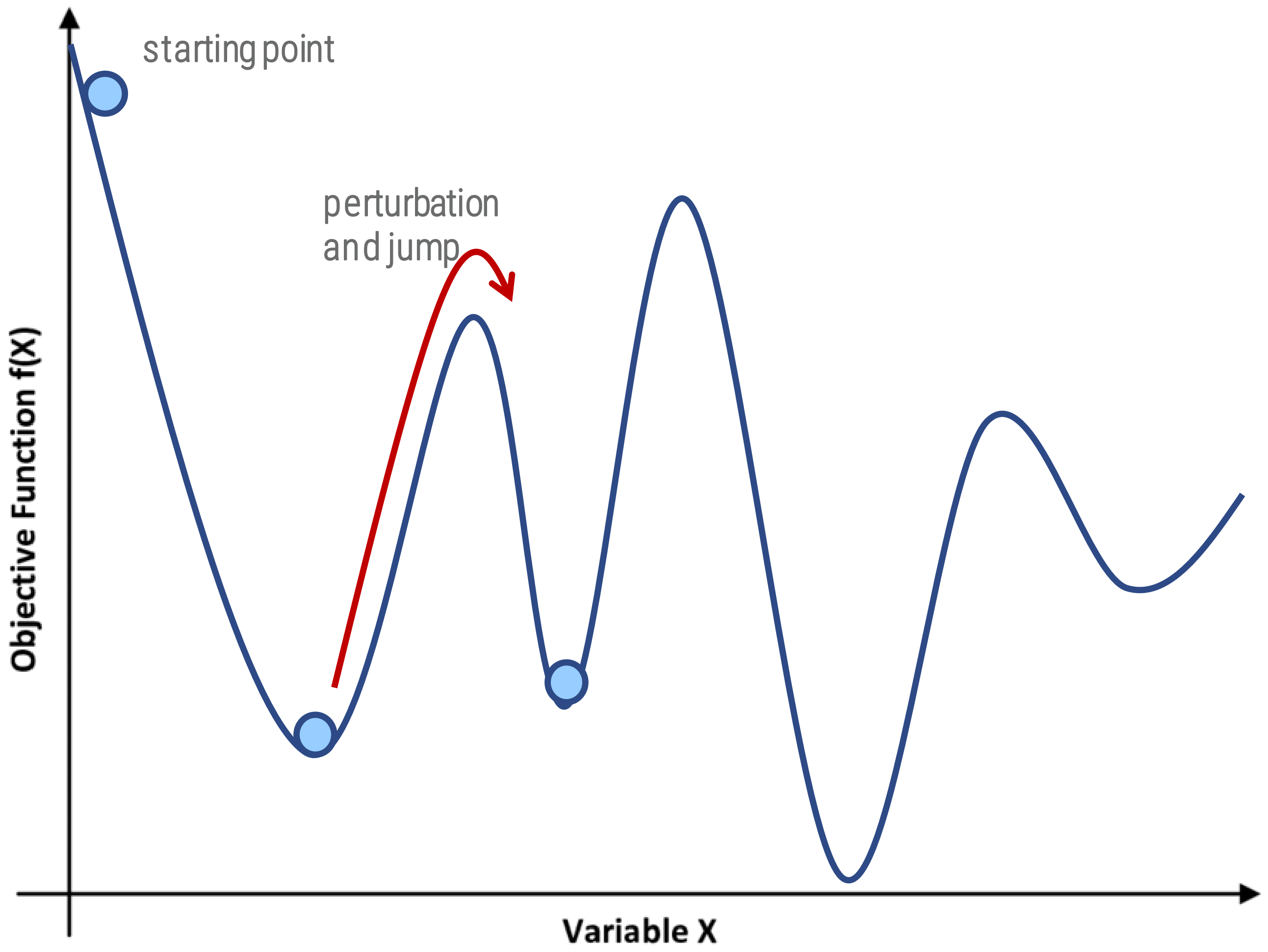


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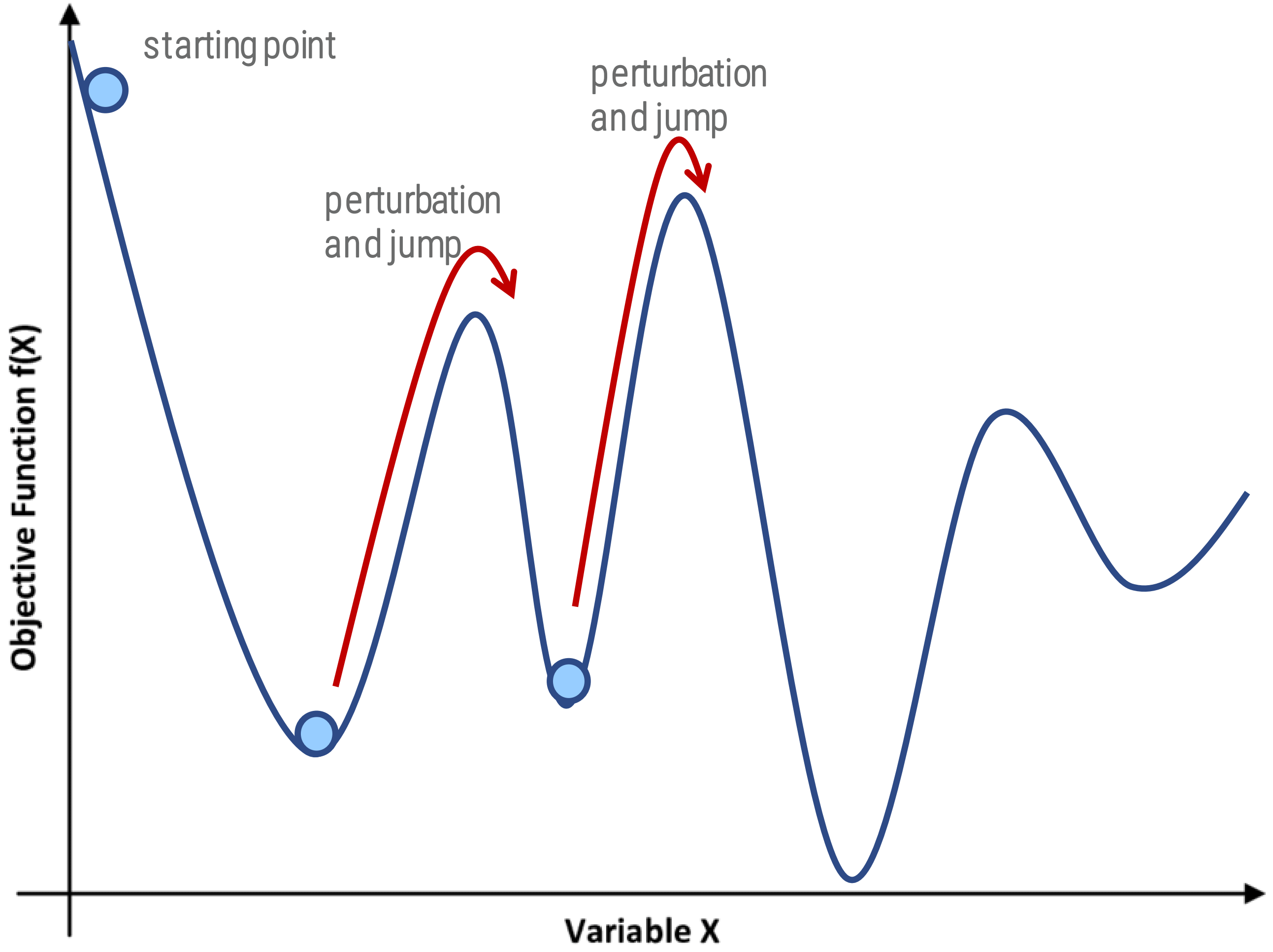

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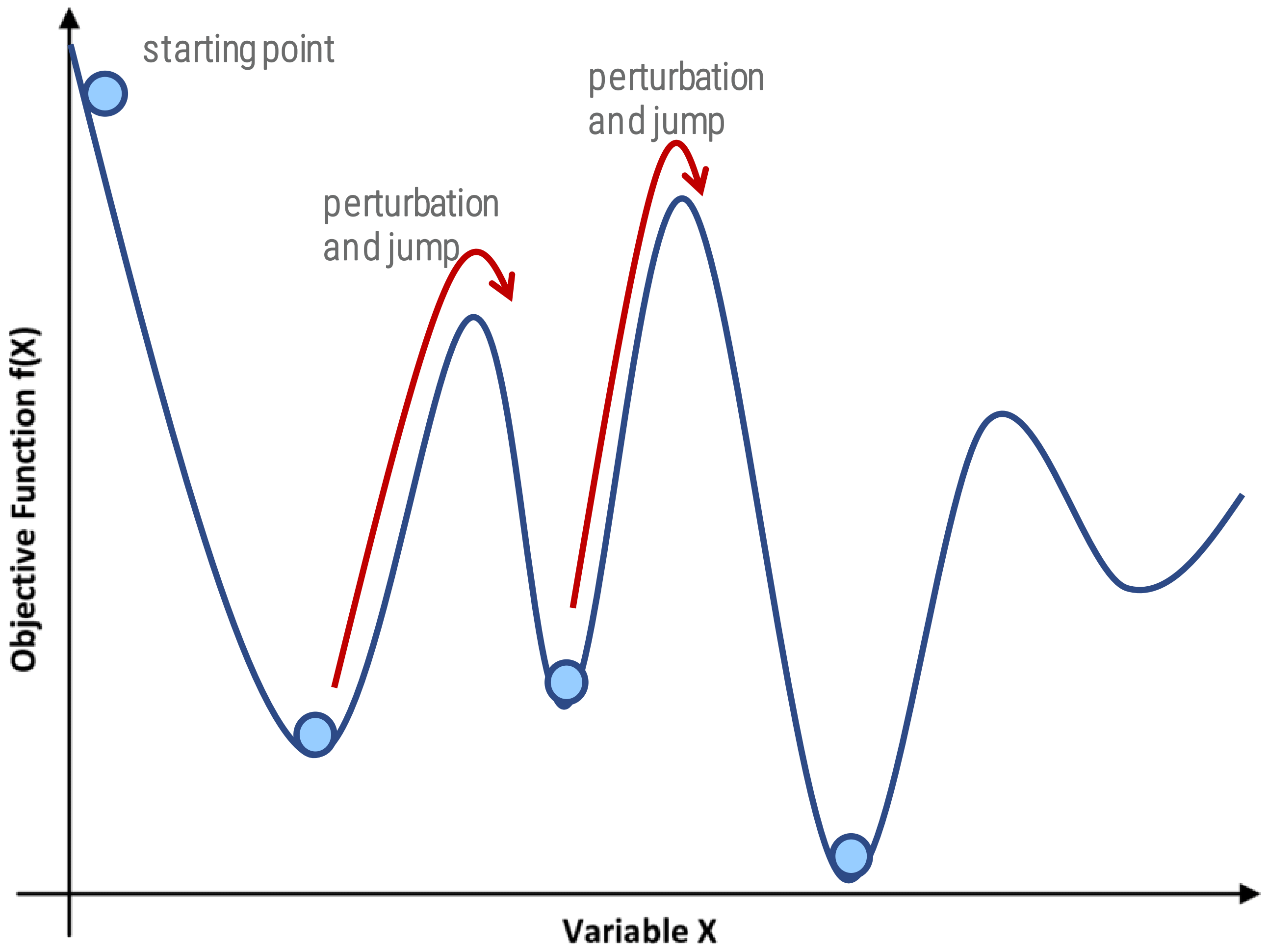

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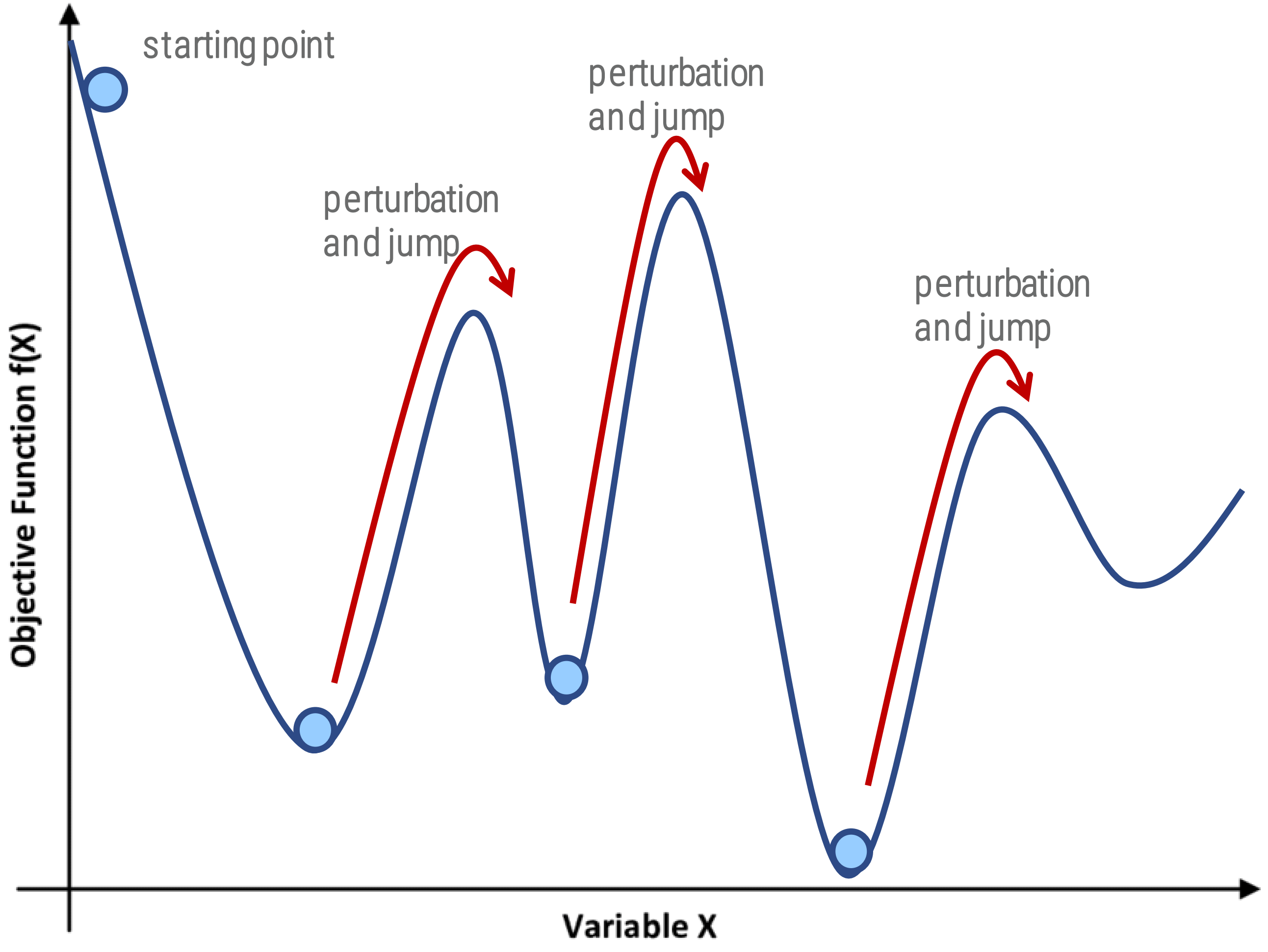

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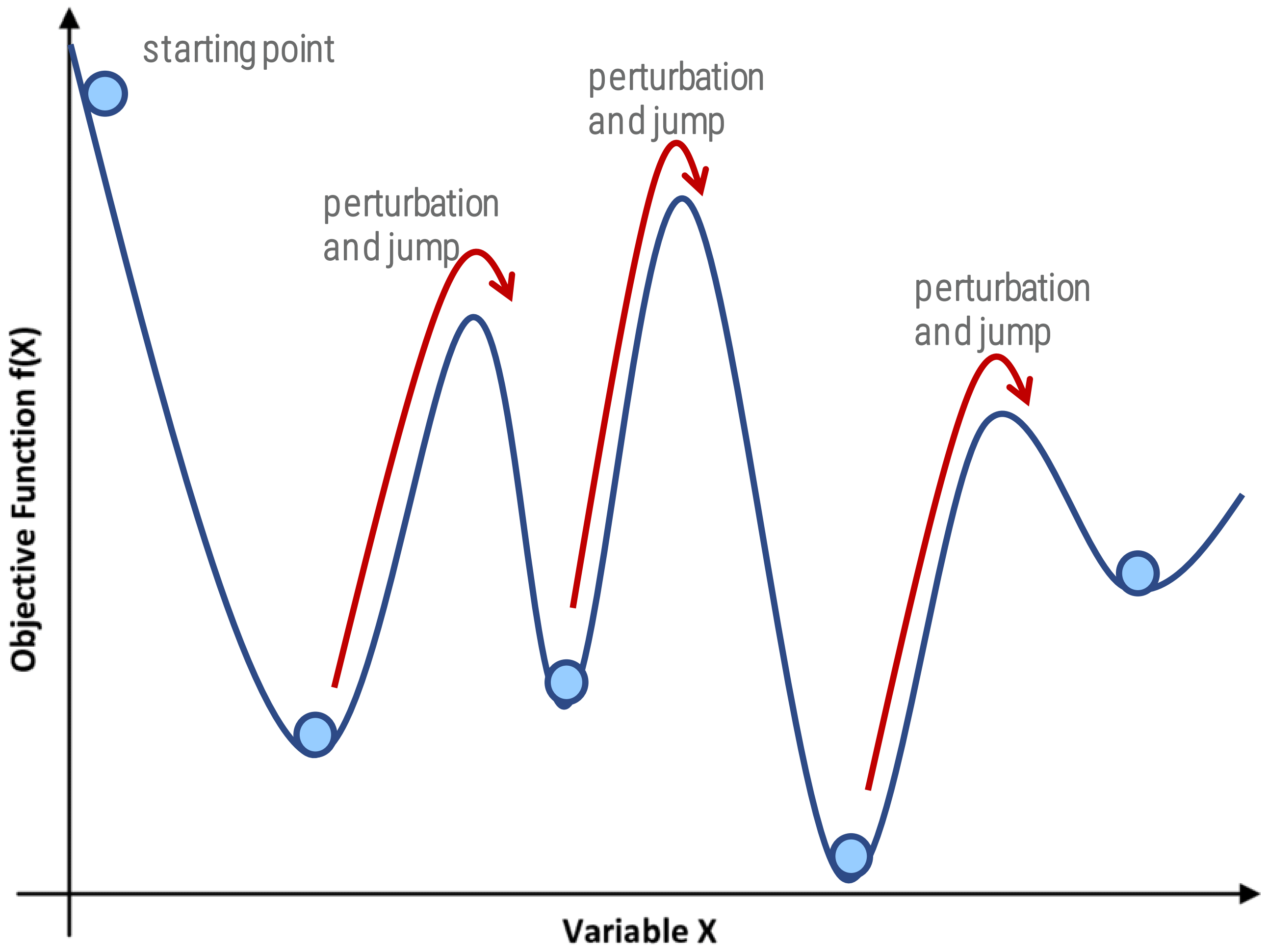

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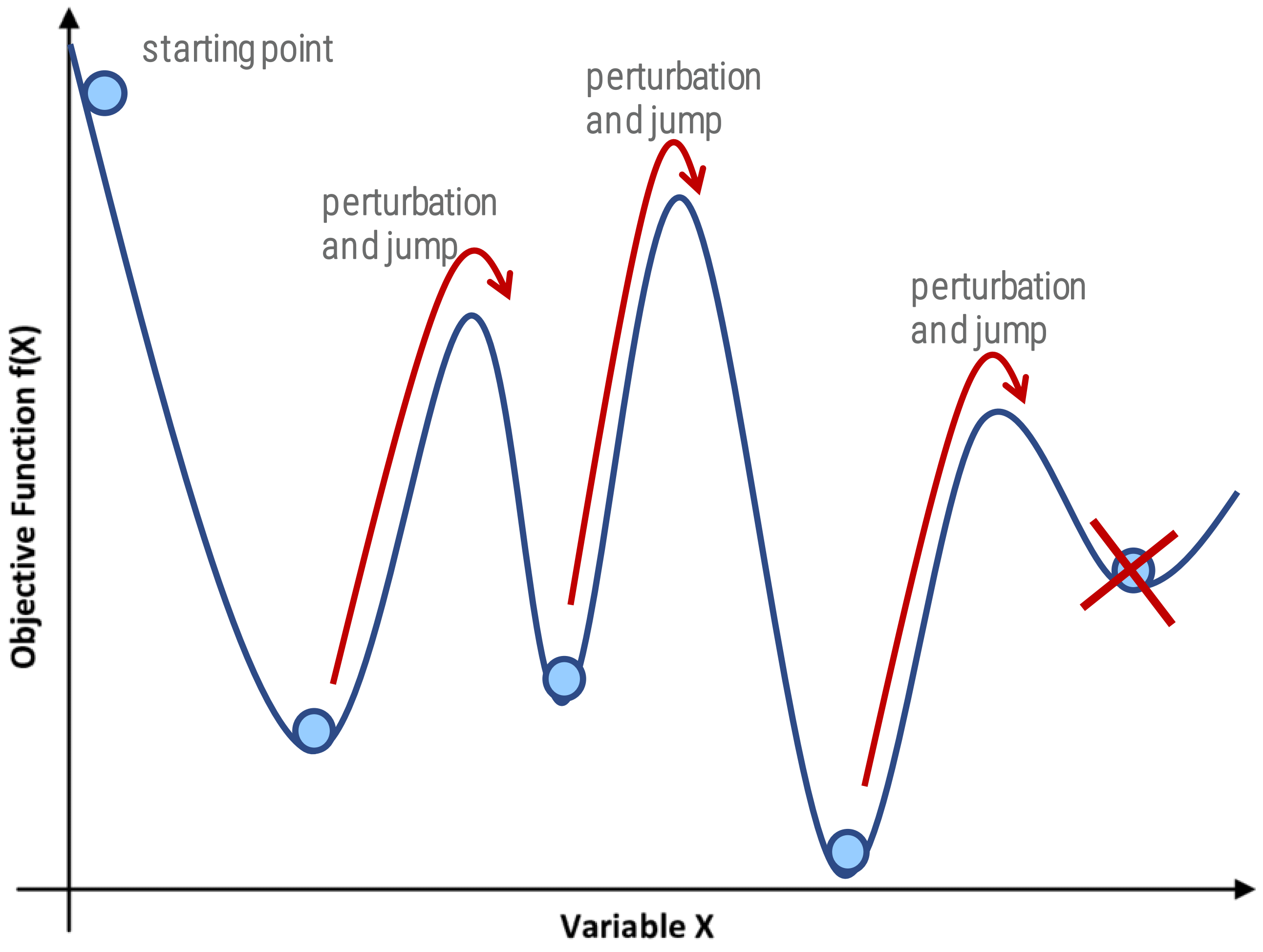

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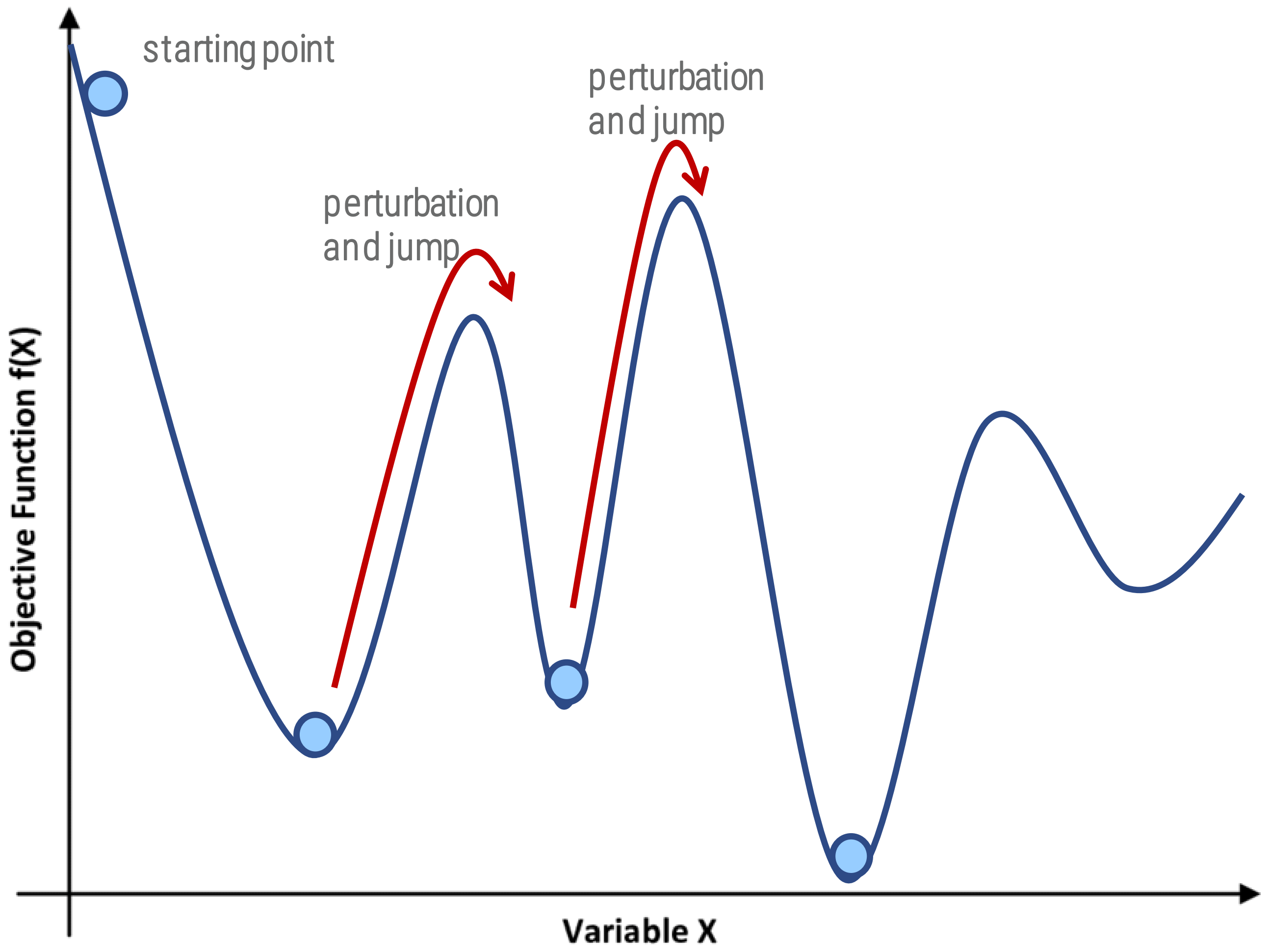
- Every time a ball lands in a hole it receives a certain amount of energy, enough to make it jump over another piece of graph, looking for deeper minima points.



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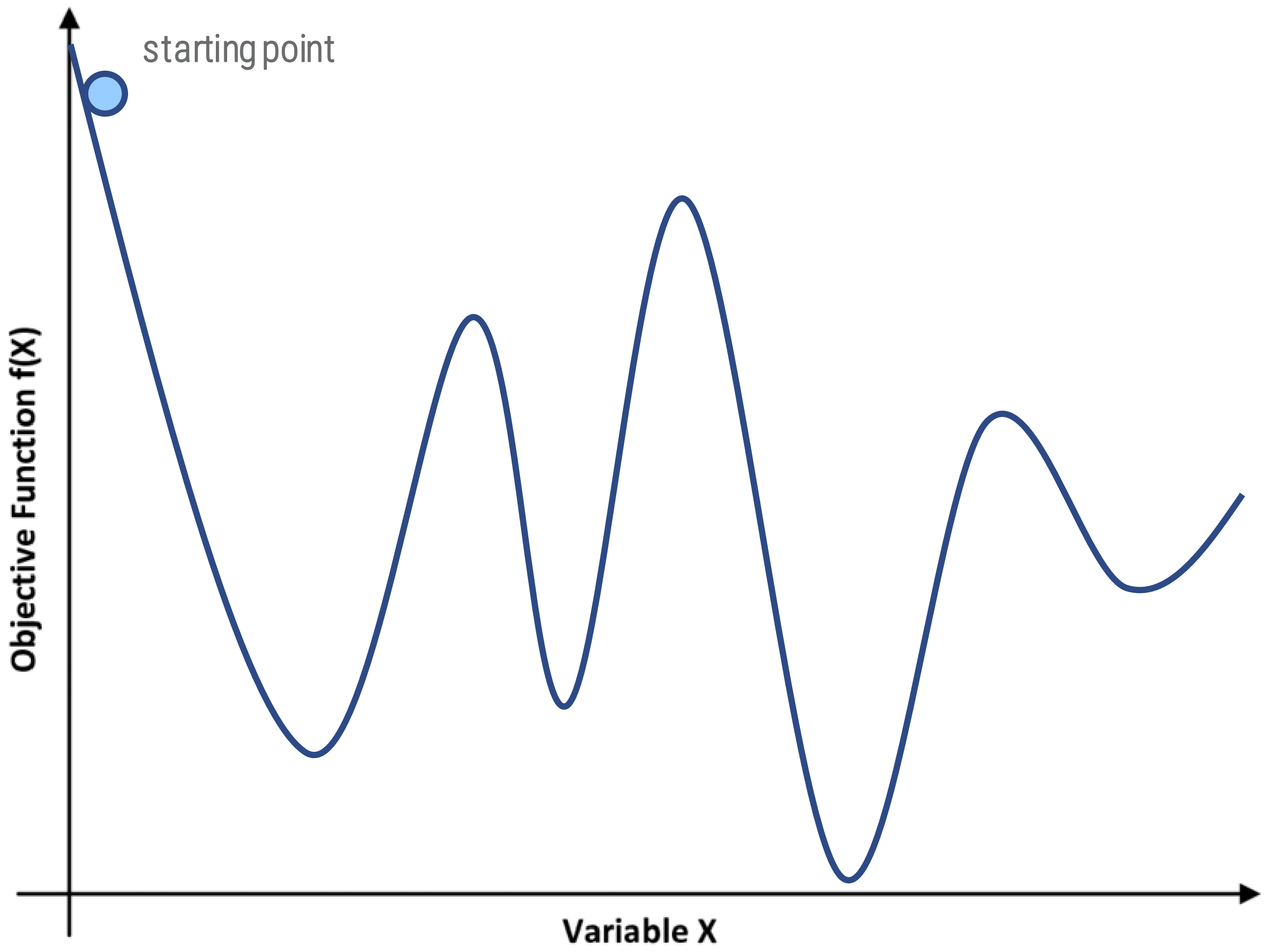

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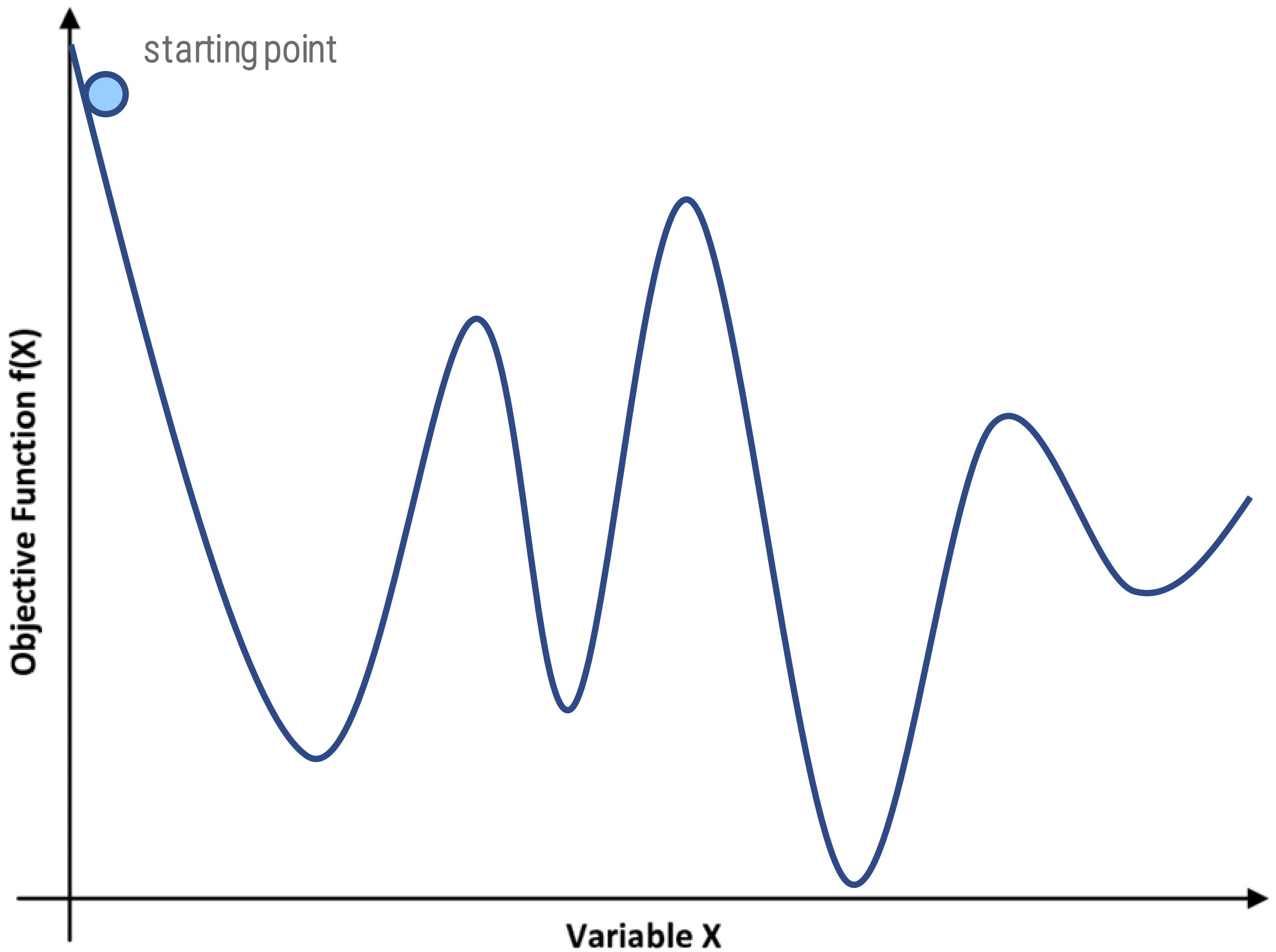

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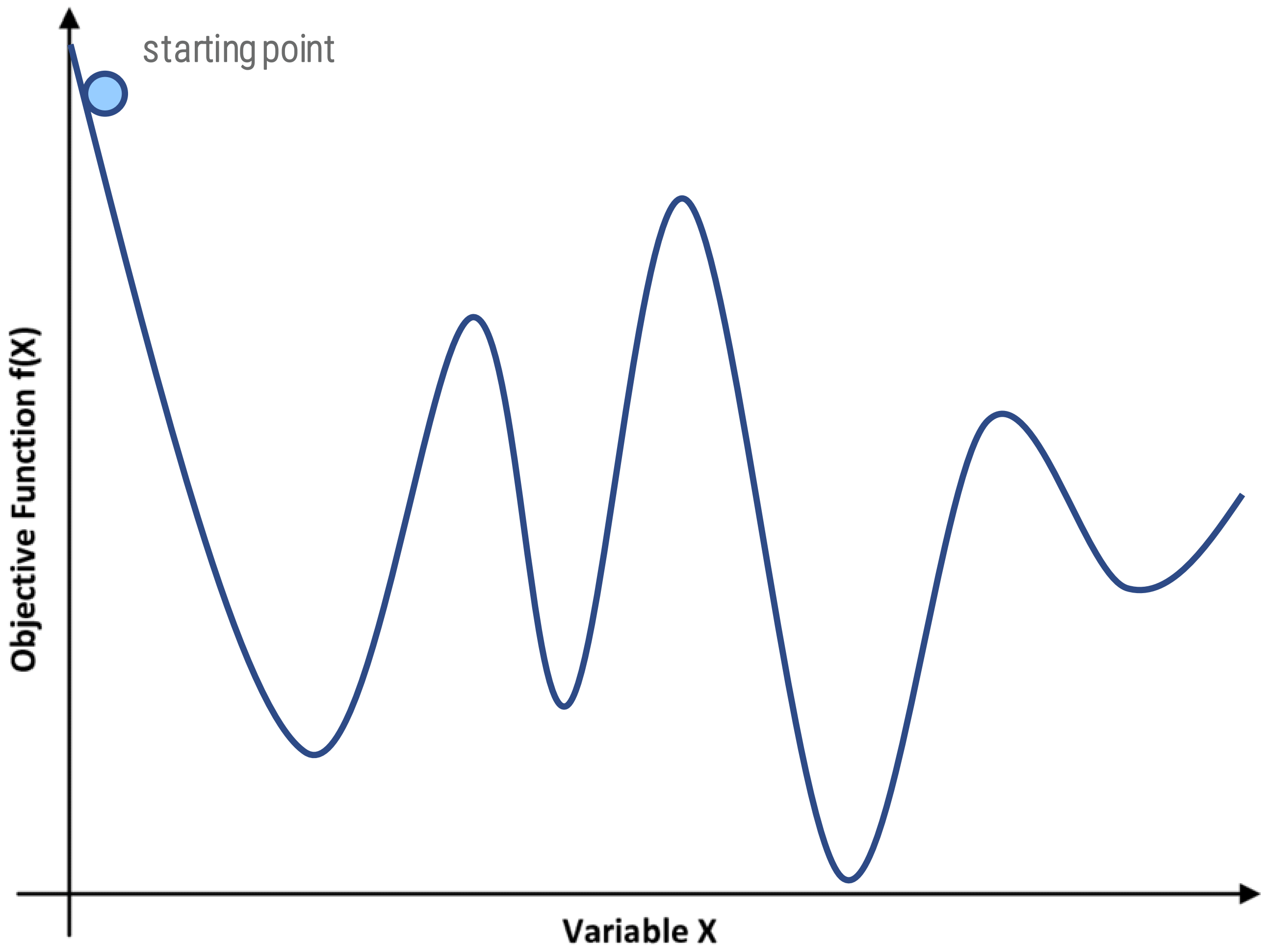

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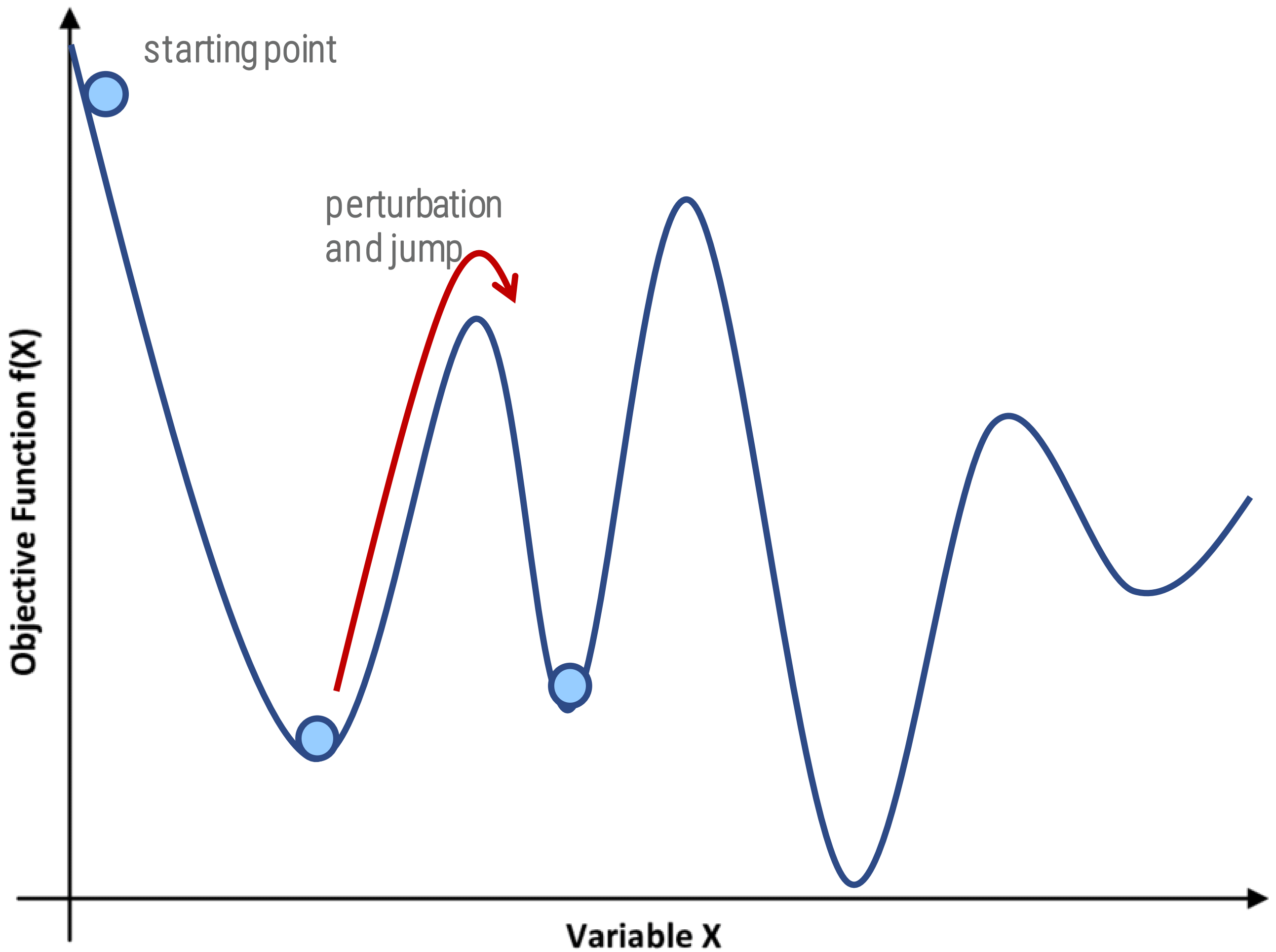

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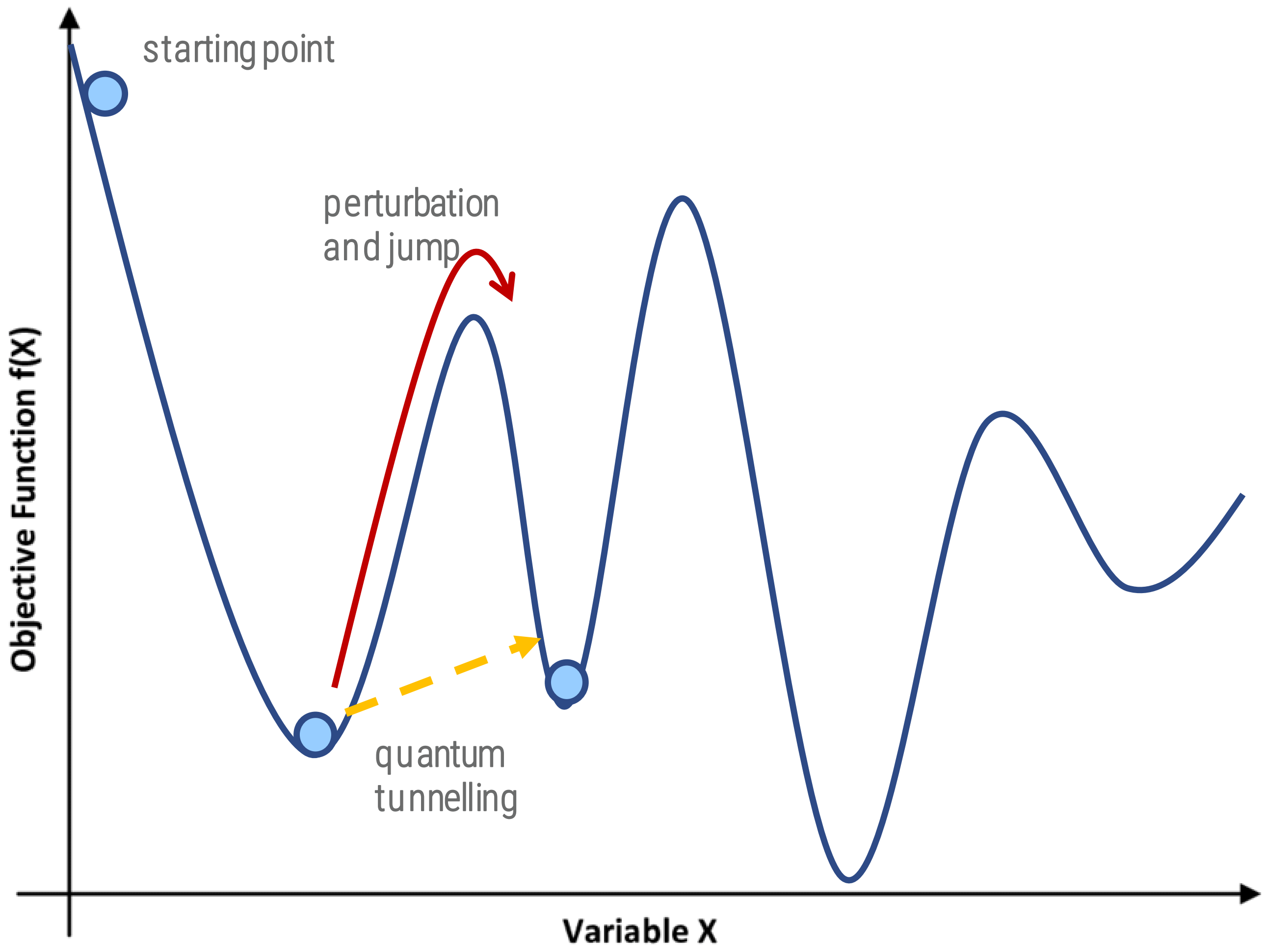

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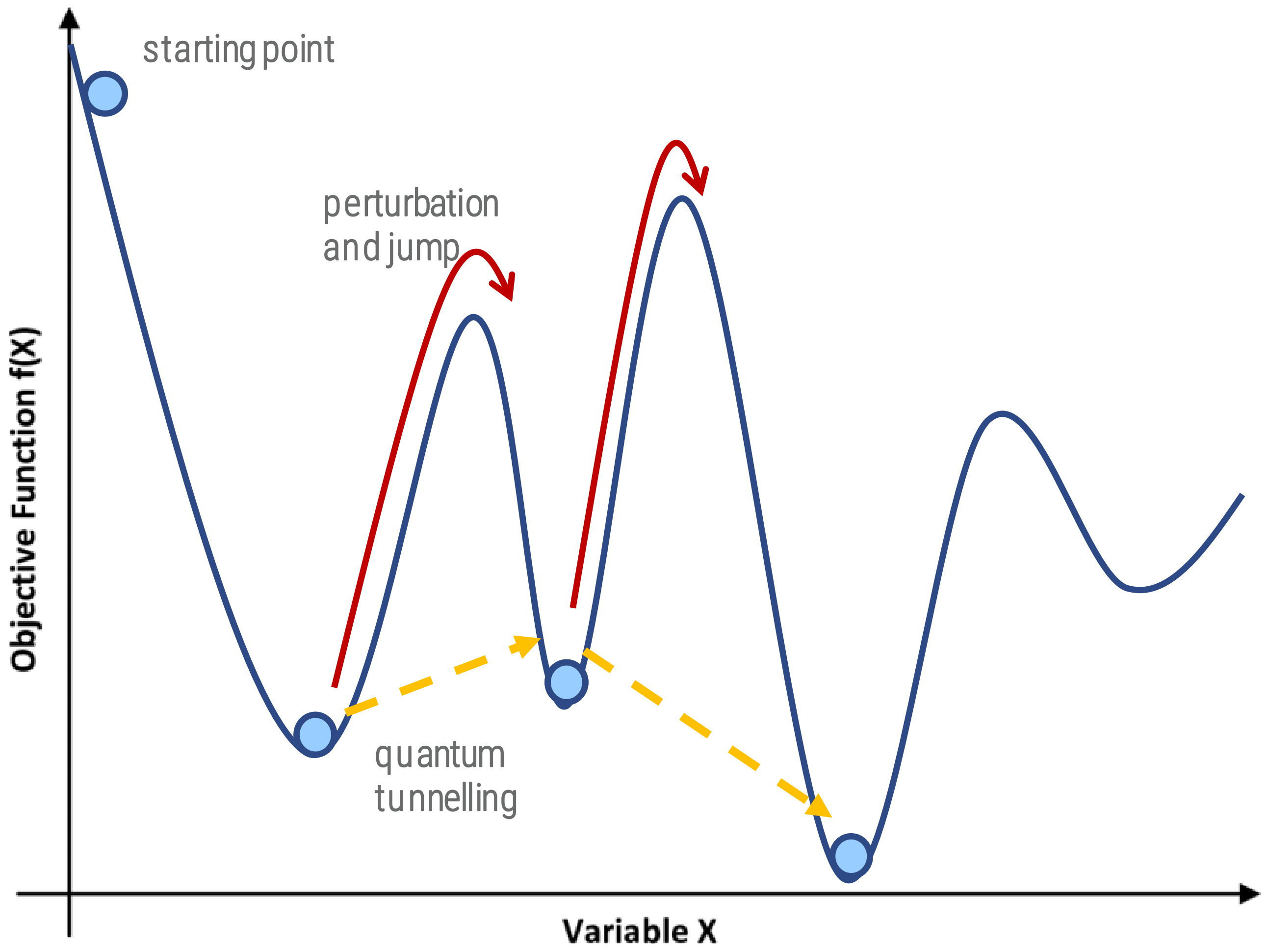

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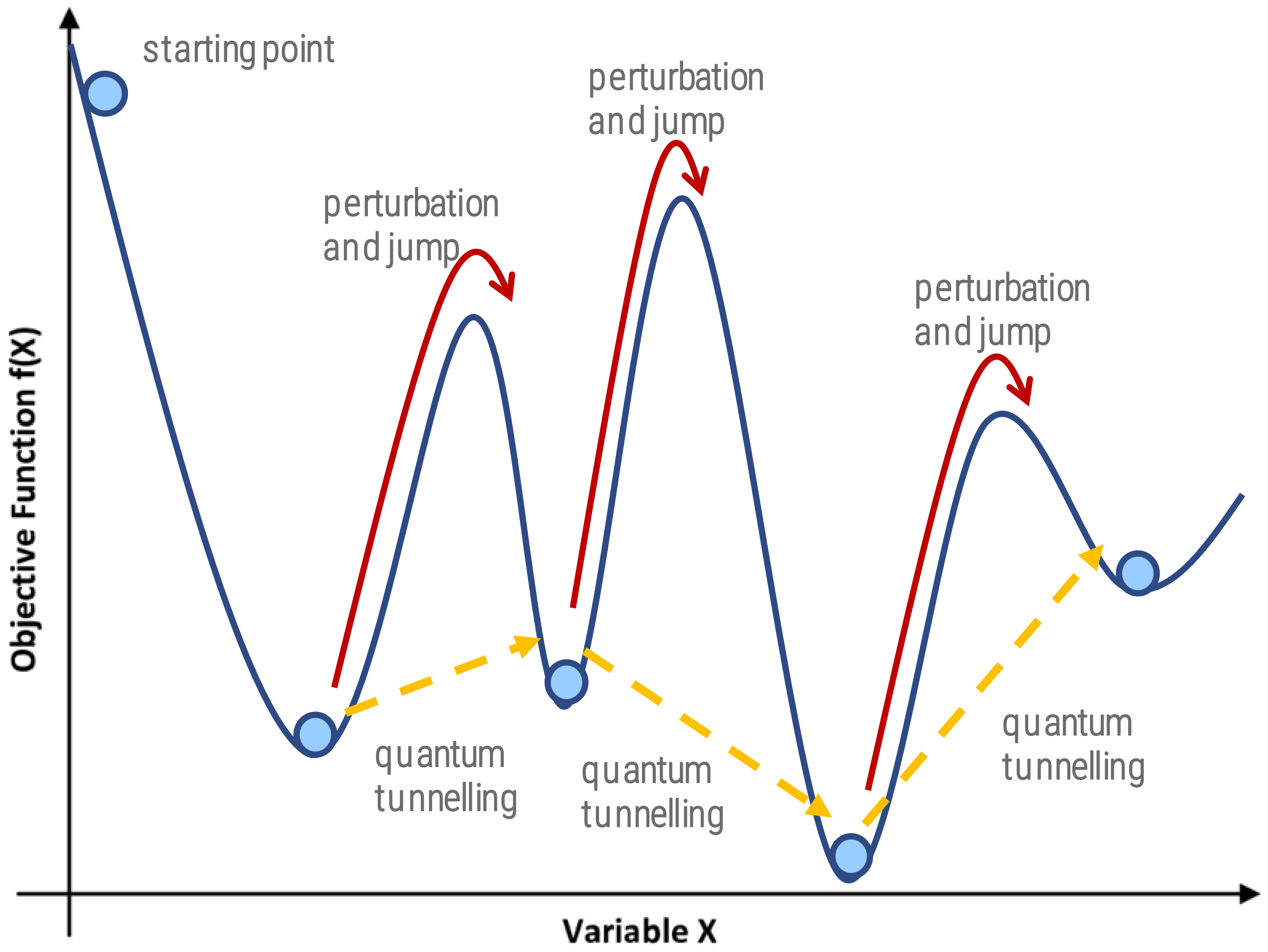

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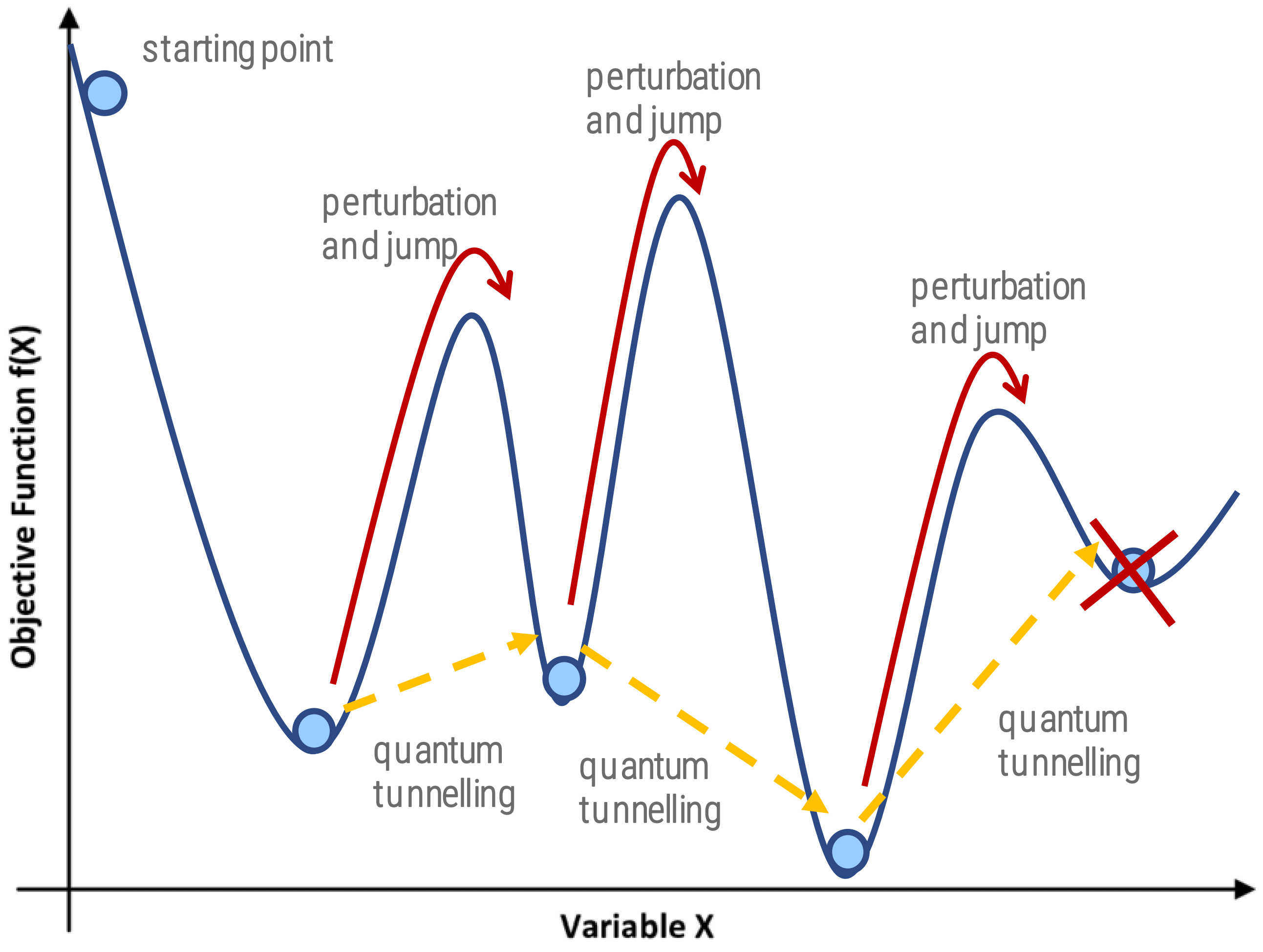
- How does the Quantum Annealing process work? The core of the algorithm is in the **Adiabatic Theorem**:
  *A physical system remains in its instantaneous eigenstate if a given perturbation is acting on it slowly enough and if there is a gap between the eigenvalue and the rest of the Hamiltonian's spectrum*



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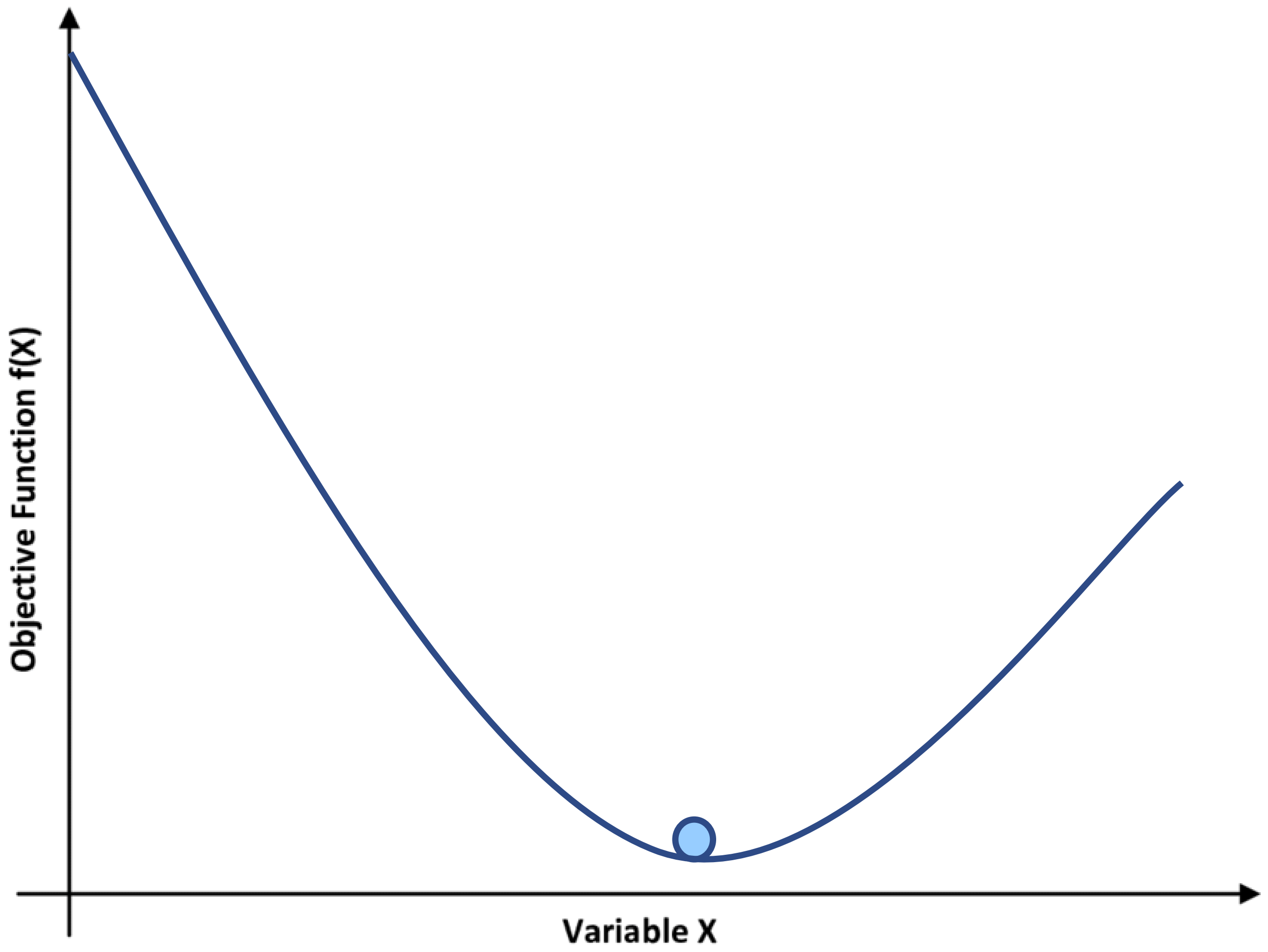

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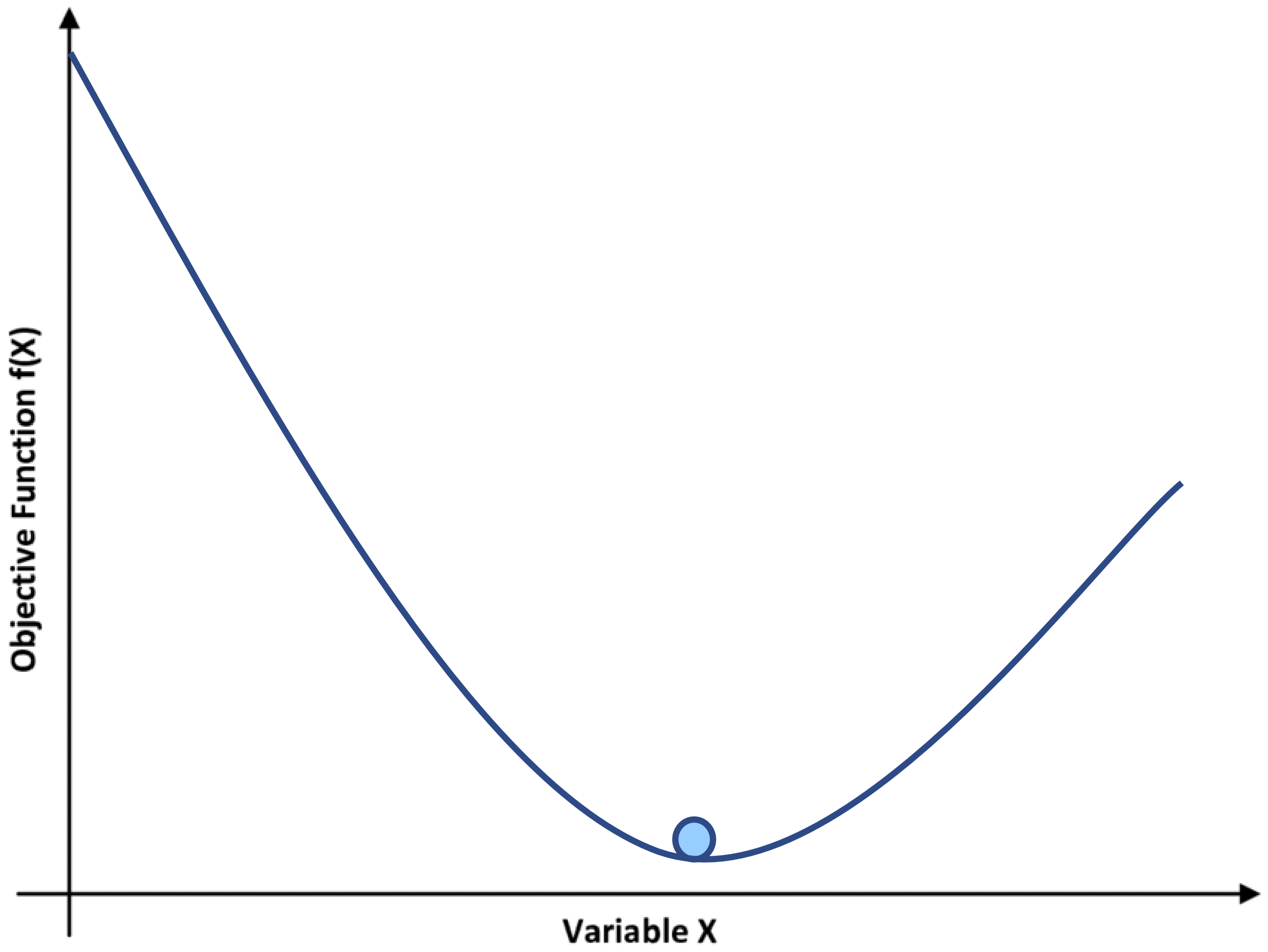

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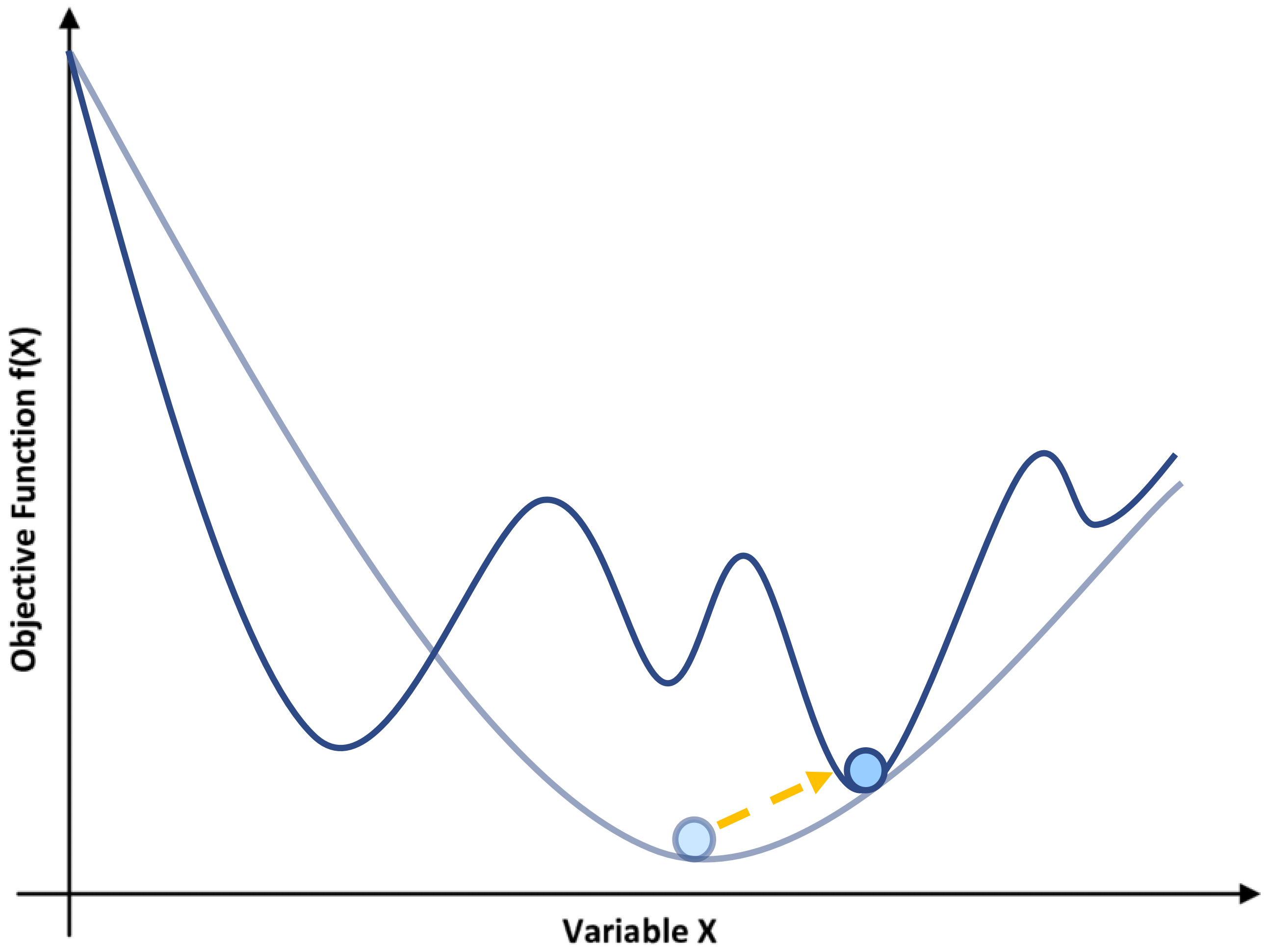

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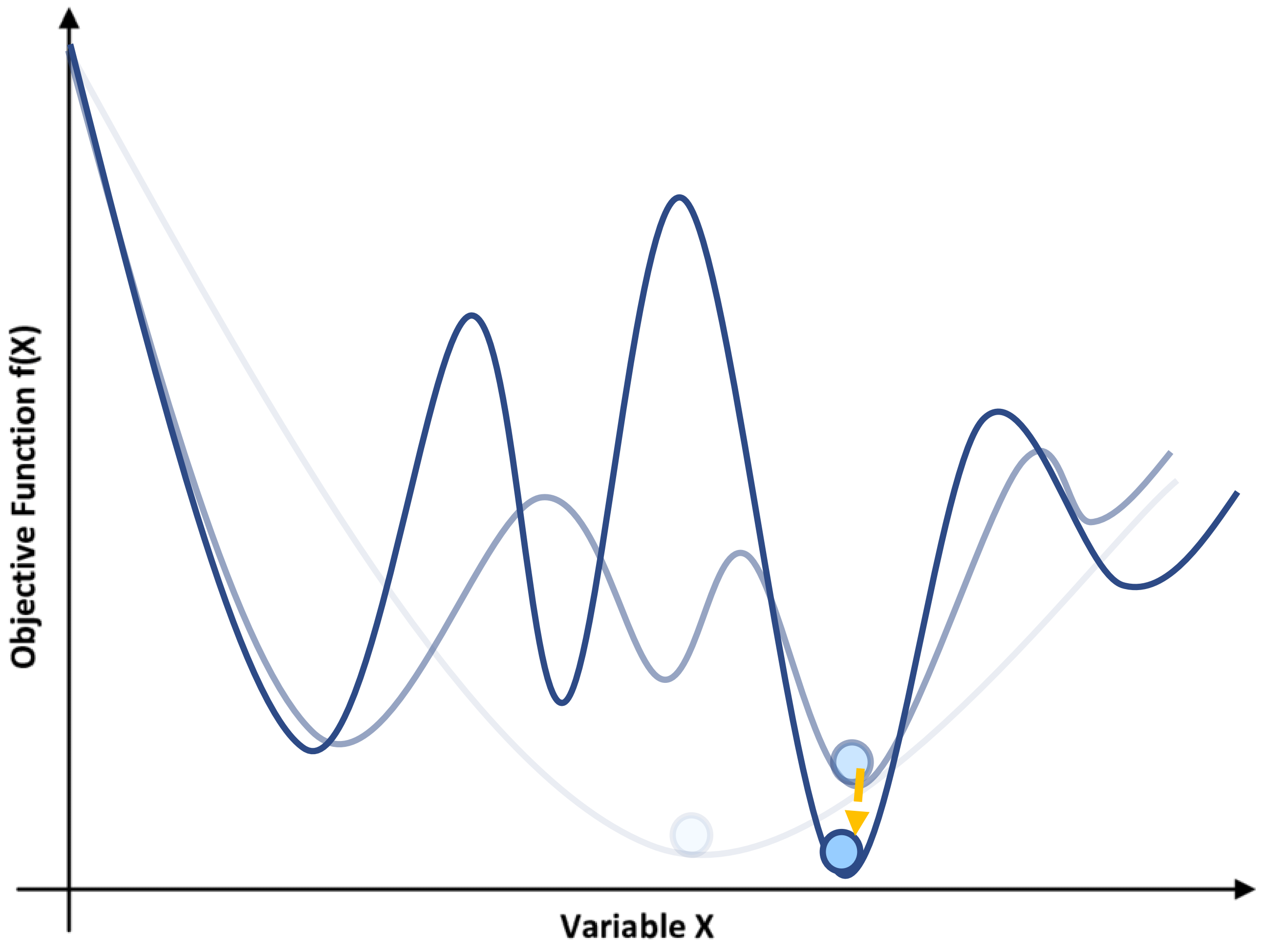

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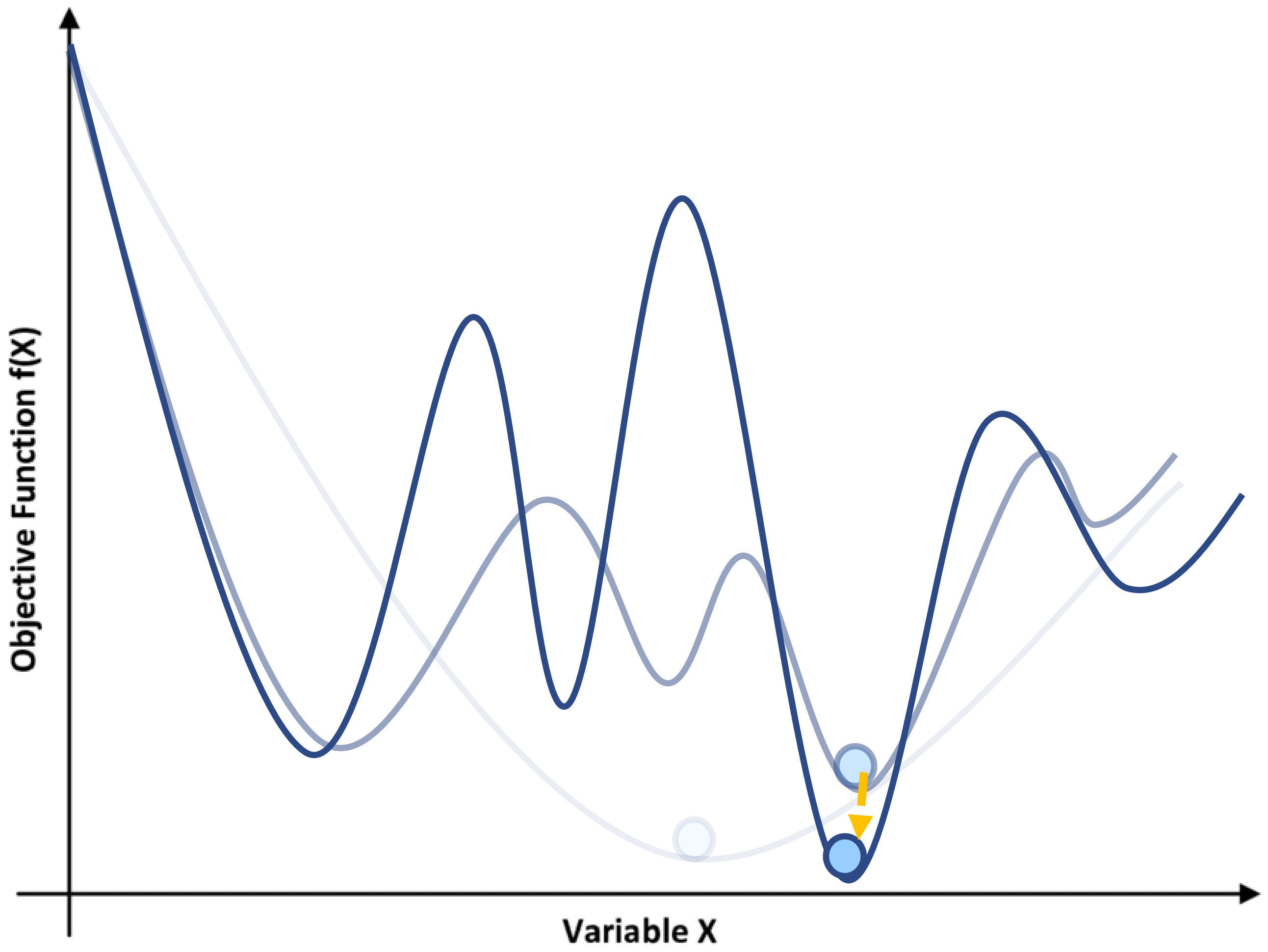

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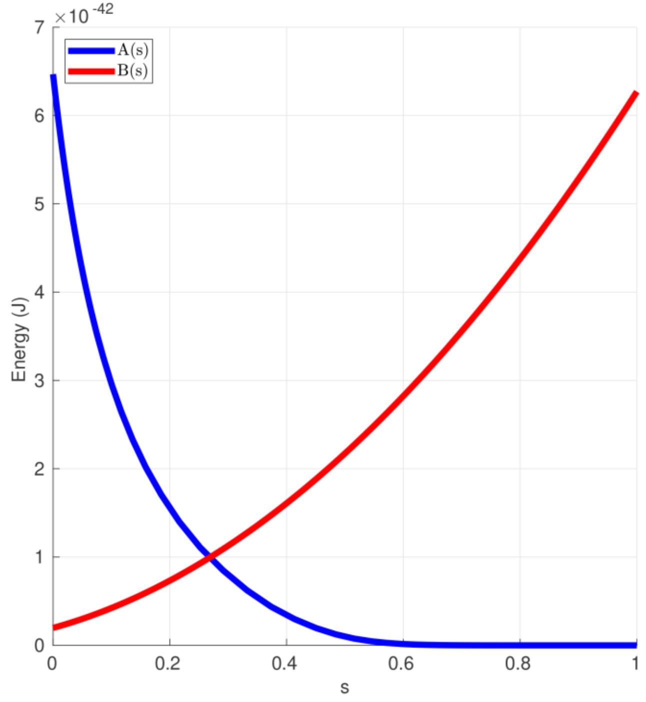


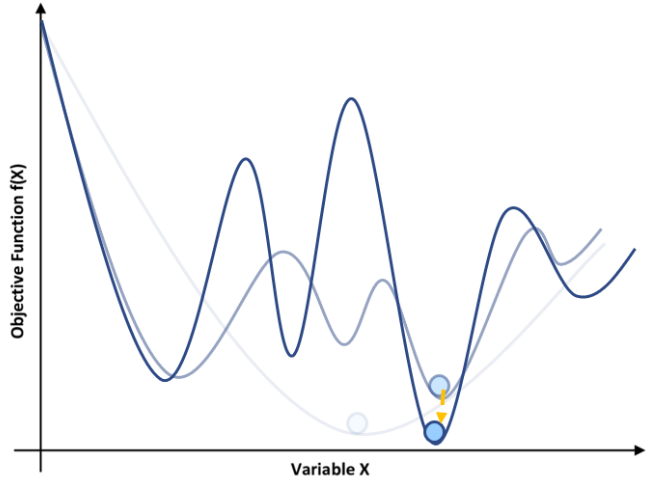

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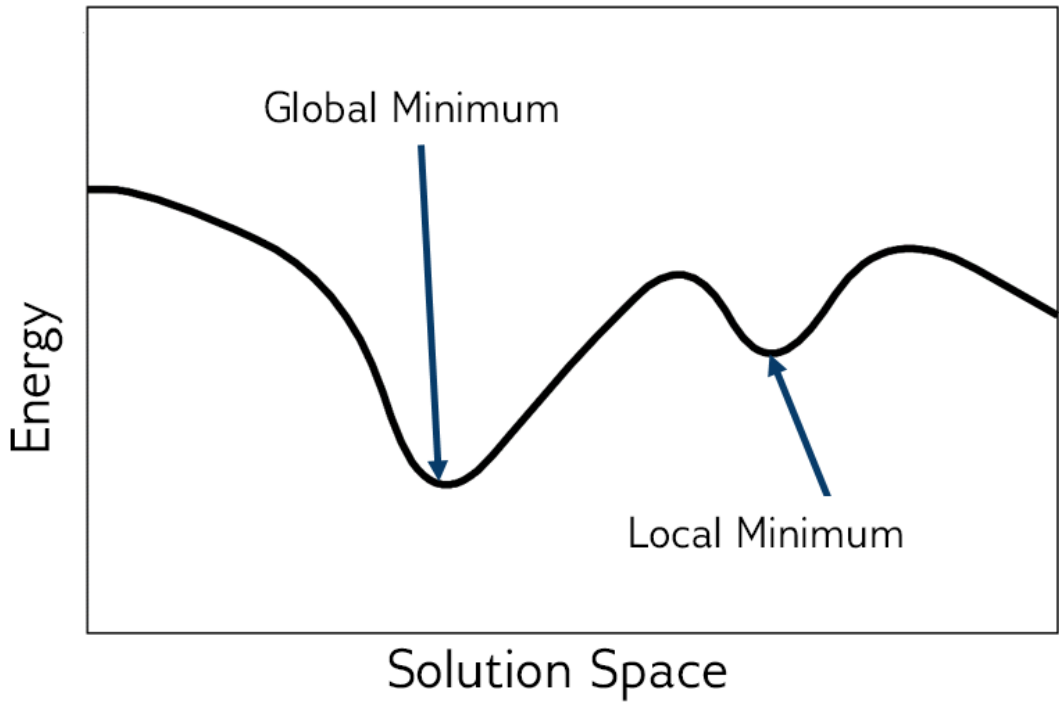
  - Graphs and embedding

# Objective Function

- To express a problem in a form that allows its resolution through quantum annealing, we need first of all an objective function



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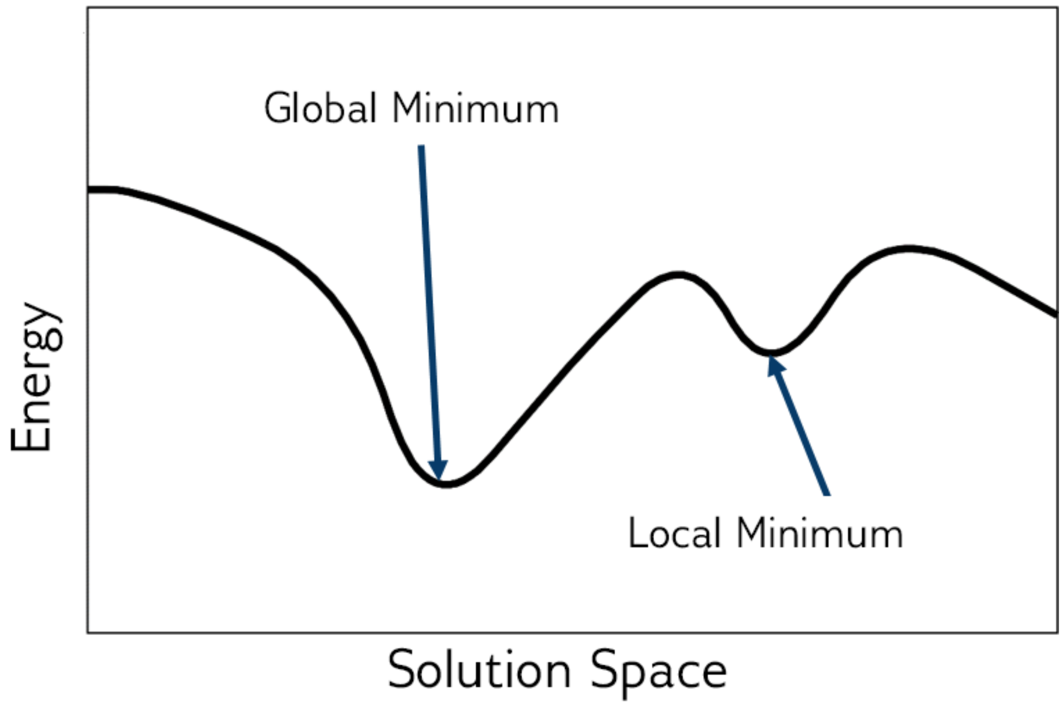

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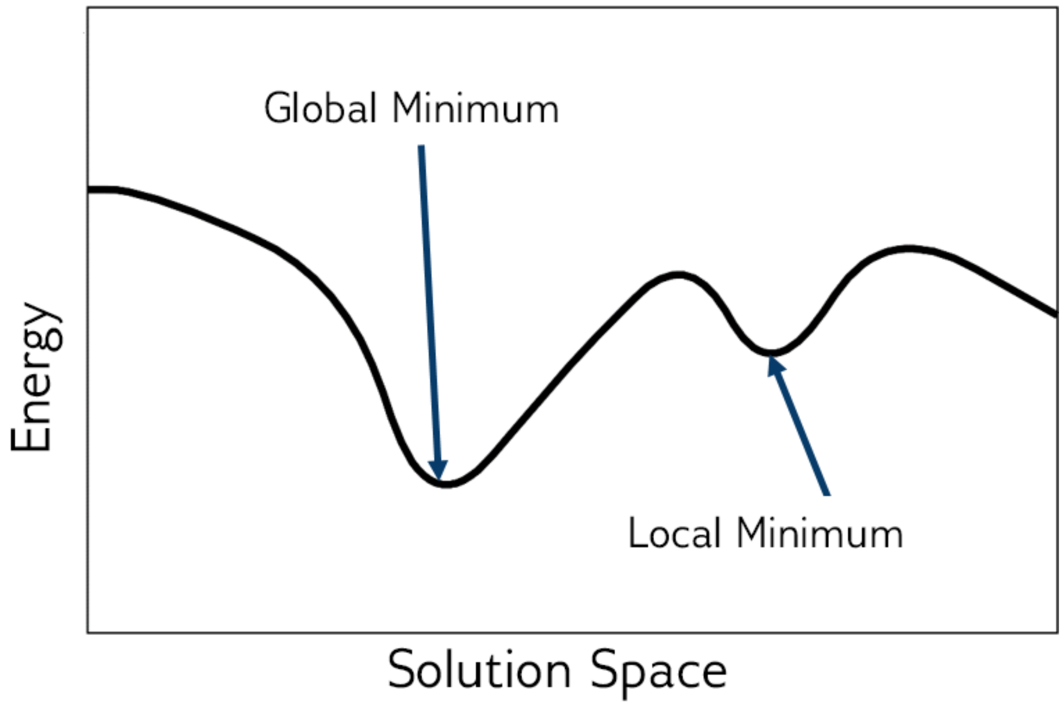

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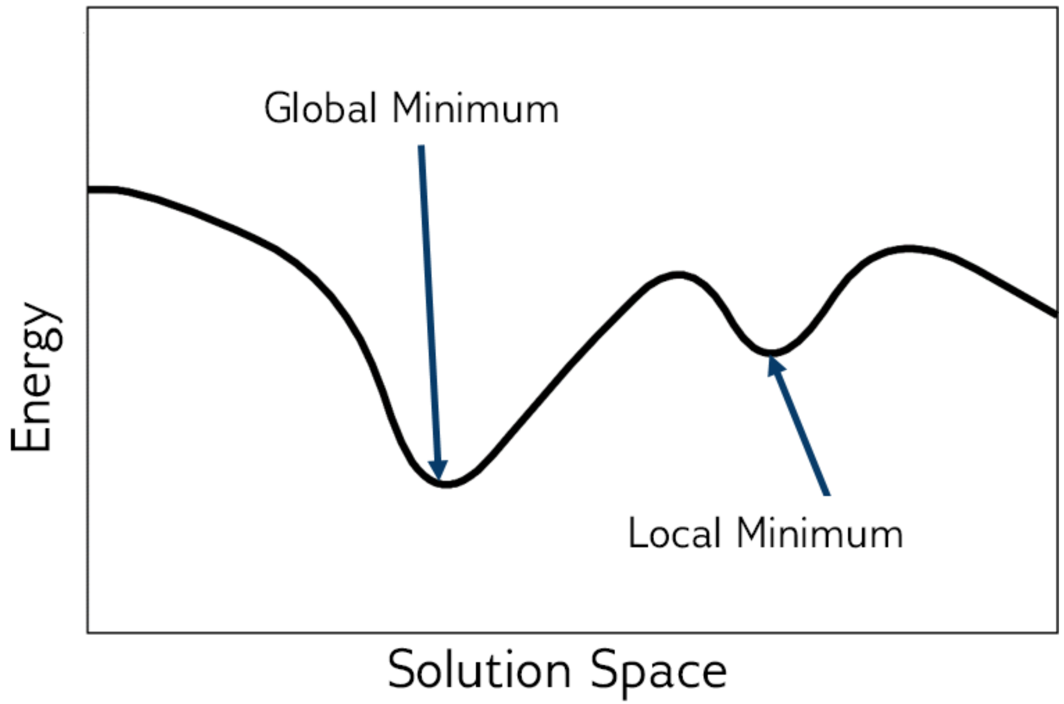
- For most problems, the lower the energy of the objective function, the better the solution. Sometimes any state of local minimum for energy is an acceptable solution to the original problem; for other problems only optimal solutions are acceptable.



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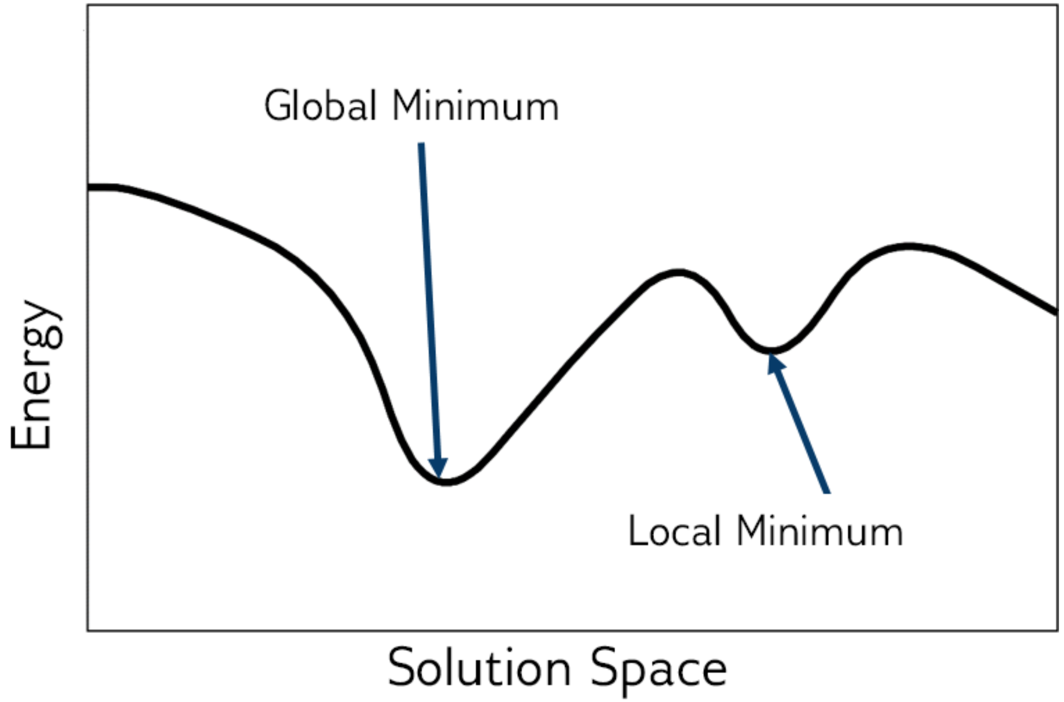


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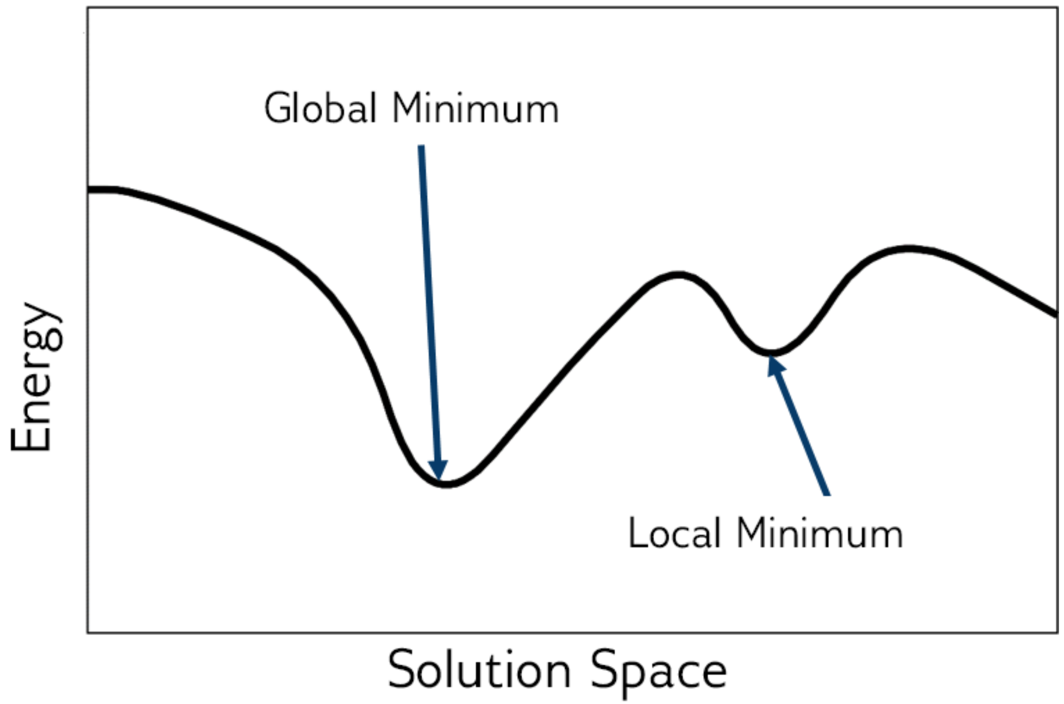


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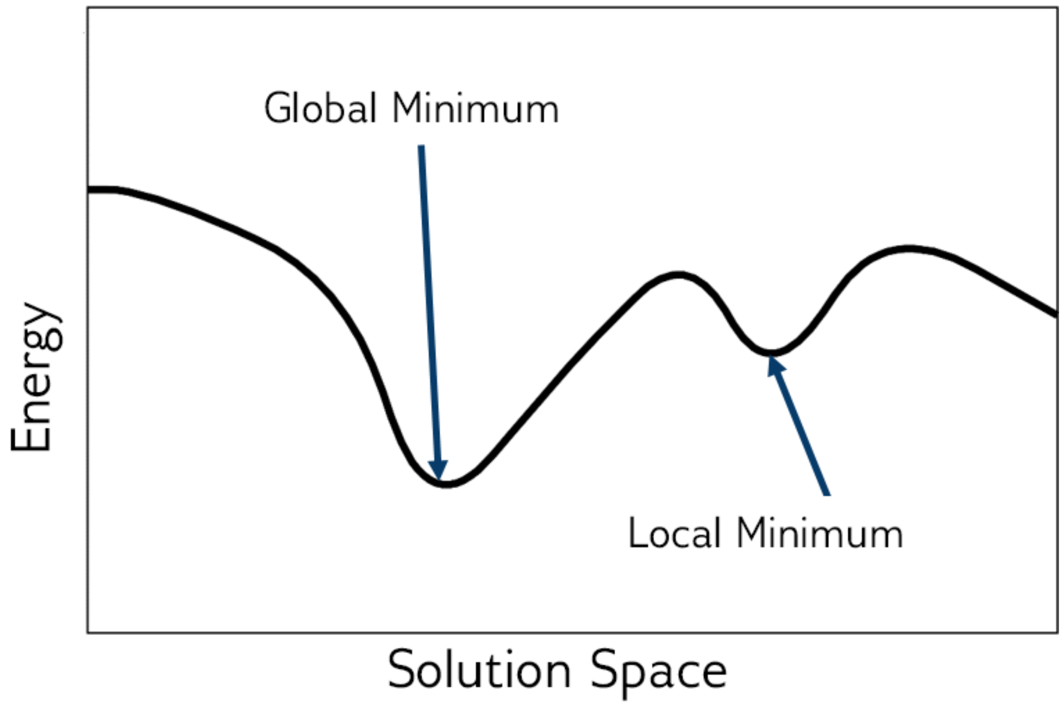


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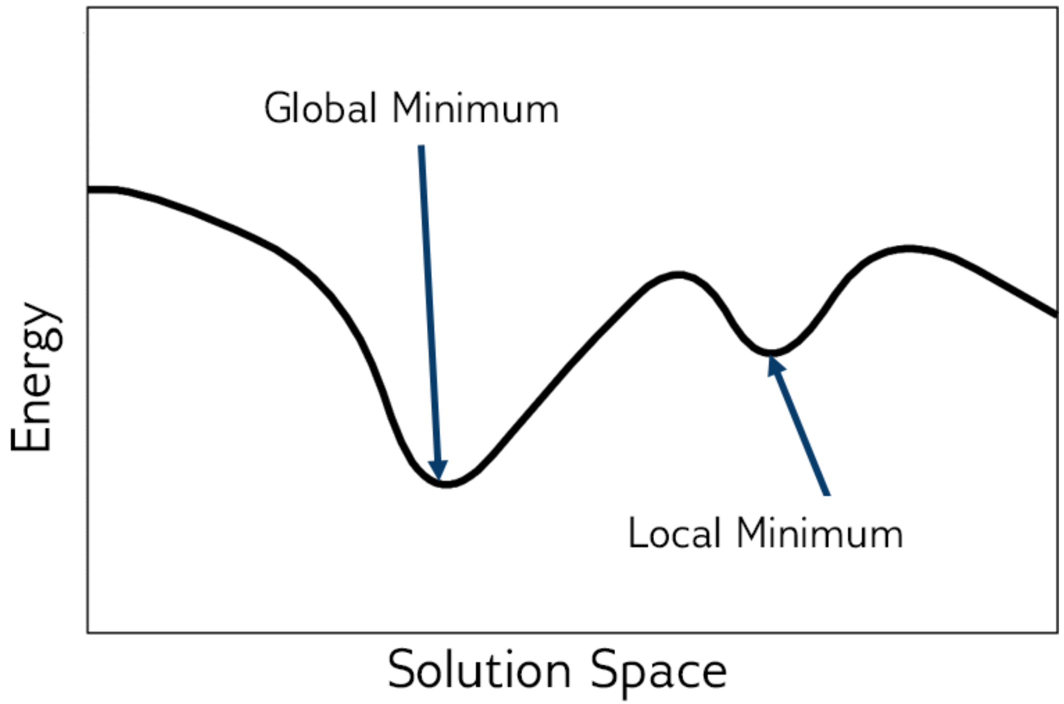
- The objective functions accepted by the quantum annealer of D-Wave are of two types (equivalent to each other): **Ising Hamiltonians and QUBO formulations**

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- Where the coefficients h represent the bias values associated with the qubits and the coefficients J represent the strength of the coupling bonds

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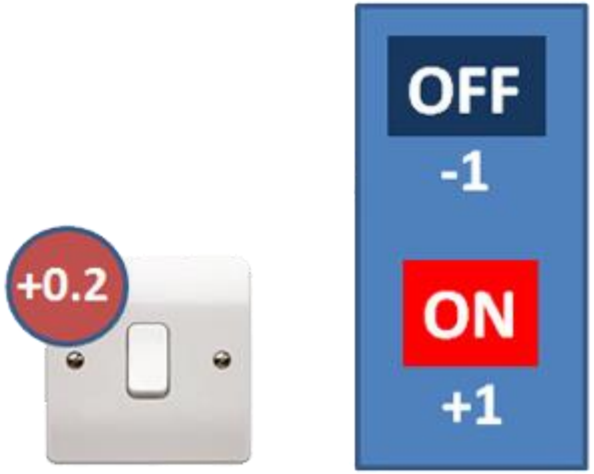


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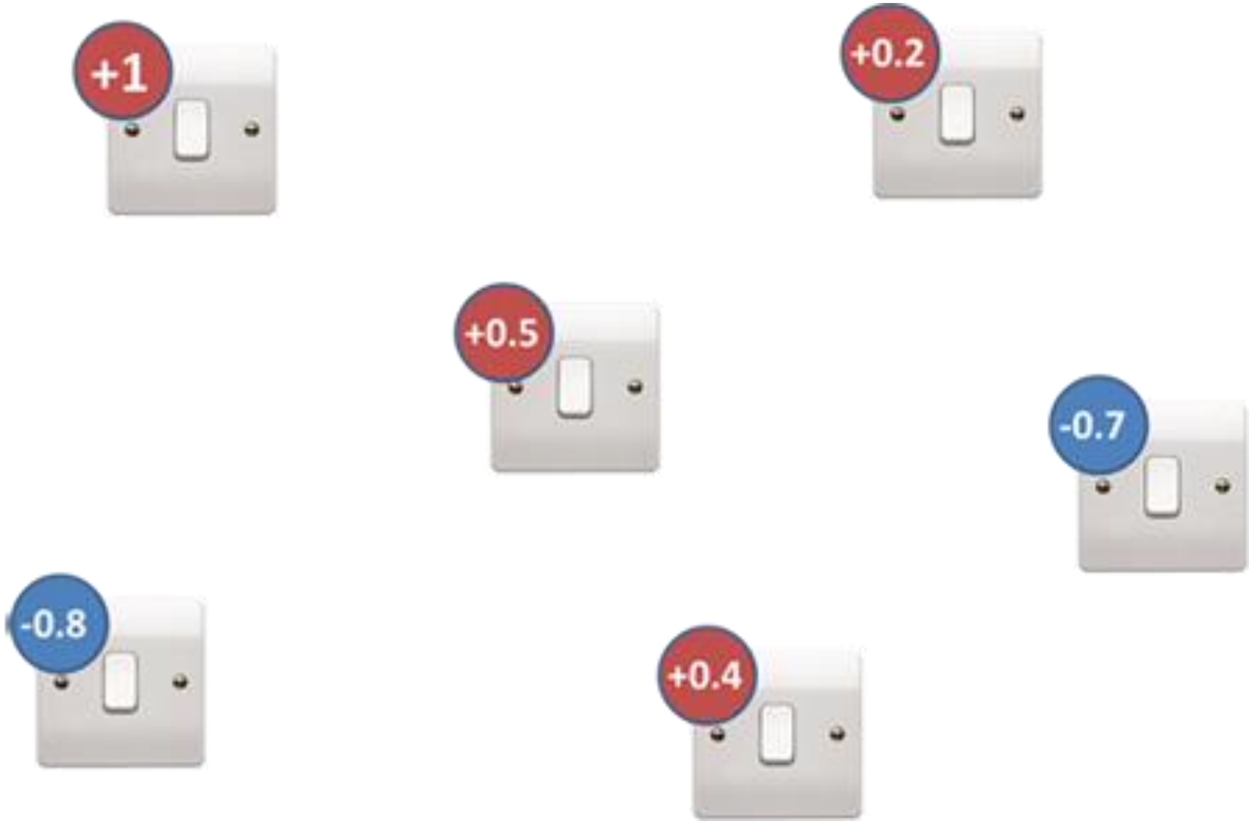


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- Furthermore, each switch has a univocally associated weight
- The value of a switch is calculated by multiplying its weight by its state
- The game consists in finding the combination of states for the switches such that the sum of their values is as low as possible

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$$E(\mathbf{s}) = \sum_i h_i s_i$$



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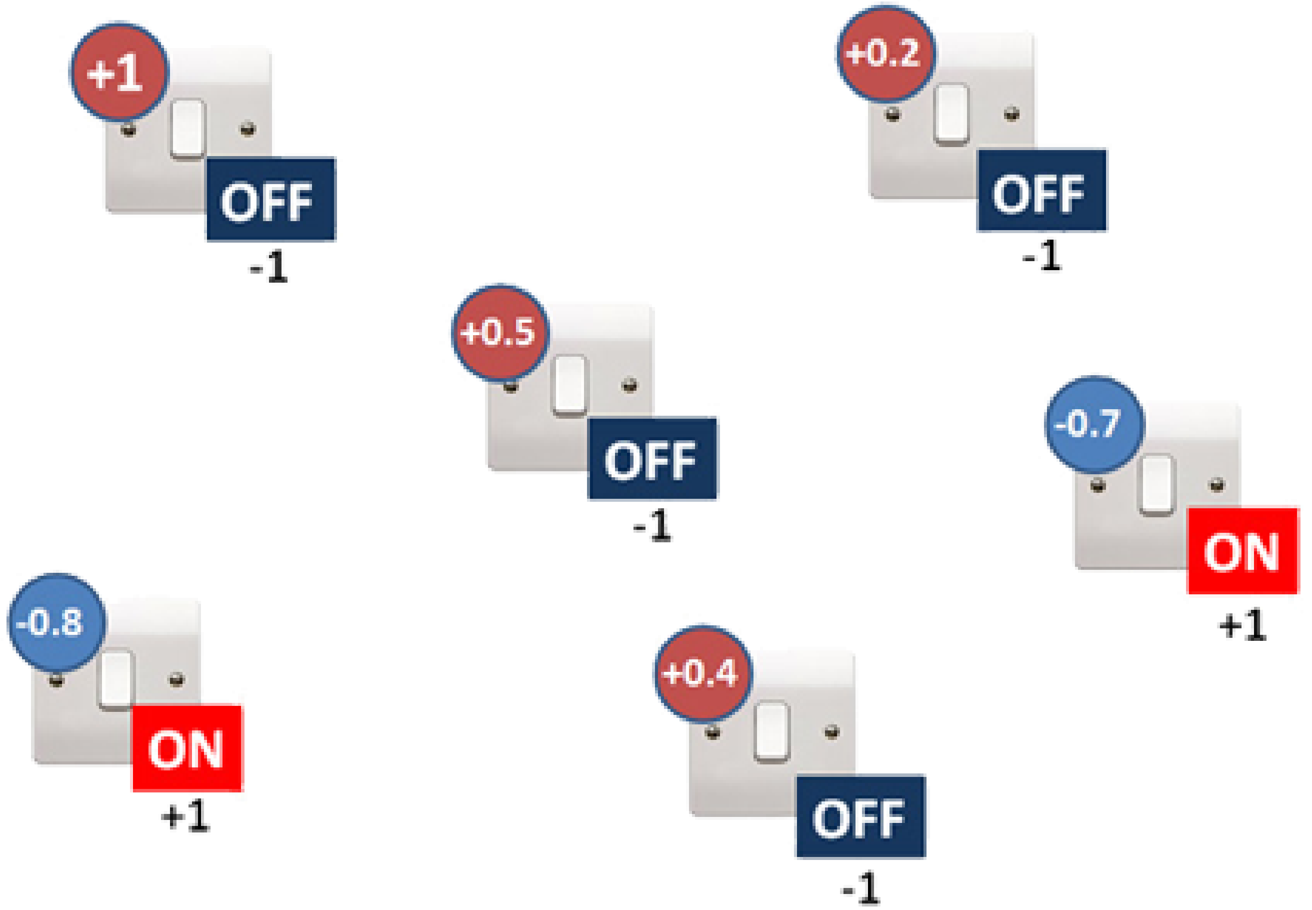



$+1 \times -1 = -1$

$+0.2 \times -1 = -0.2$

$+0.5 \times -1 = -0.5$

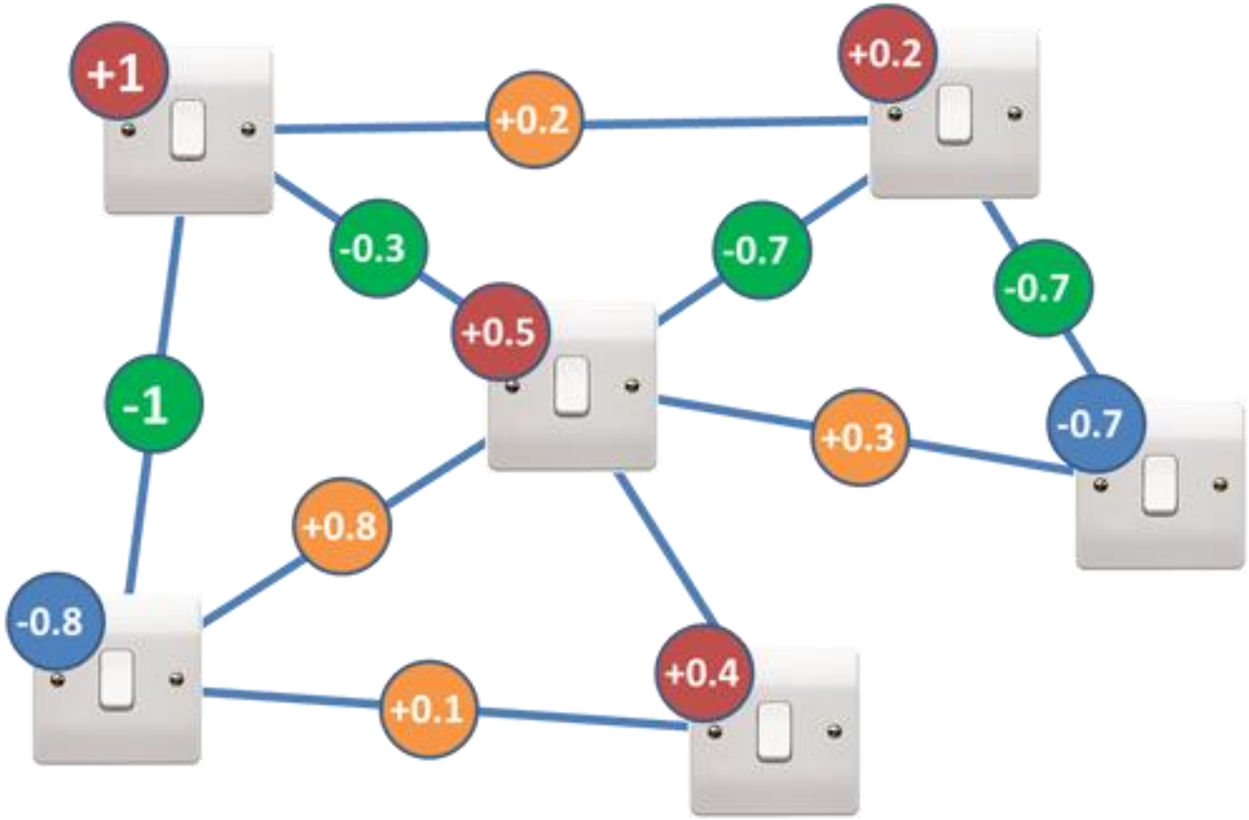
$-0.8 \times +1 = -0.8$

$+0.4 \times -1 = -0.4$

$-0.7 \times +1 = -0.7$

Total: -3.6

# Game of Switches

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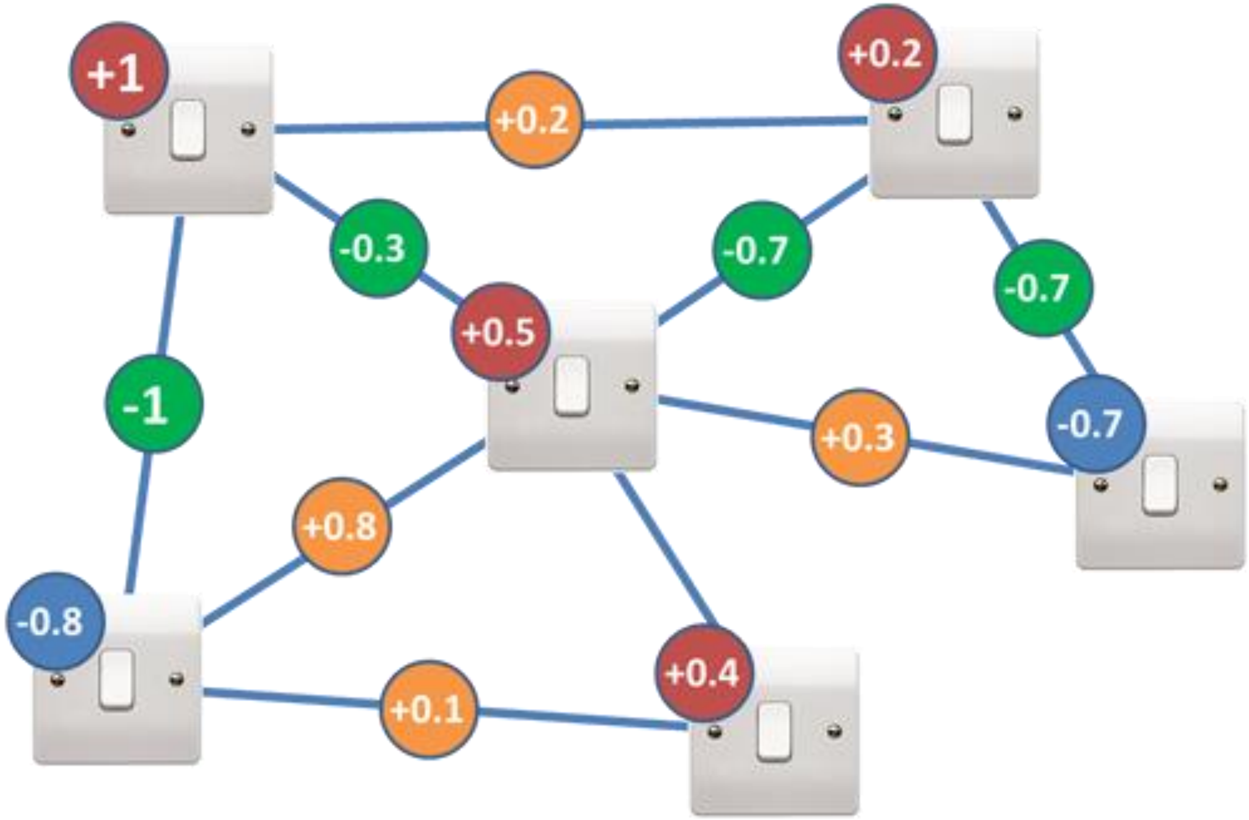


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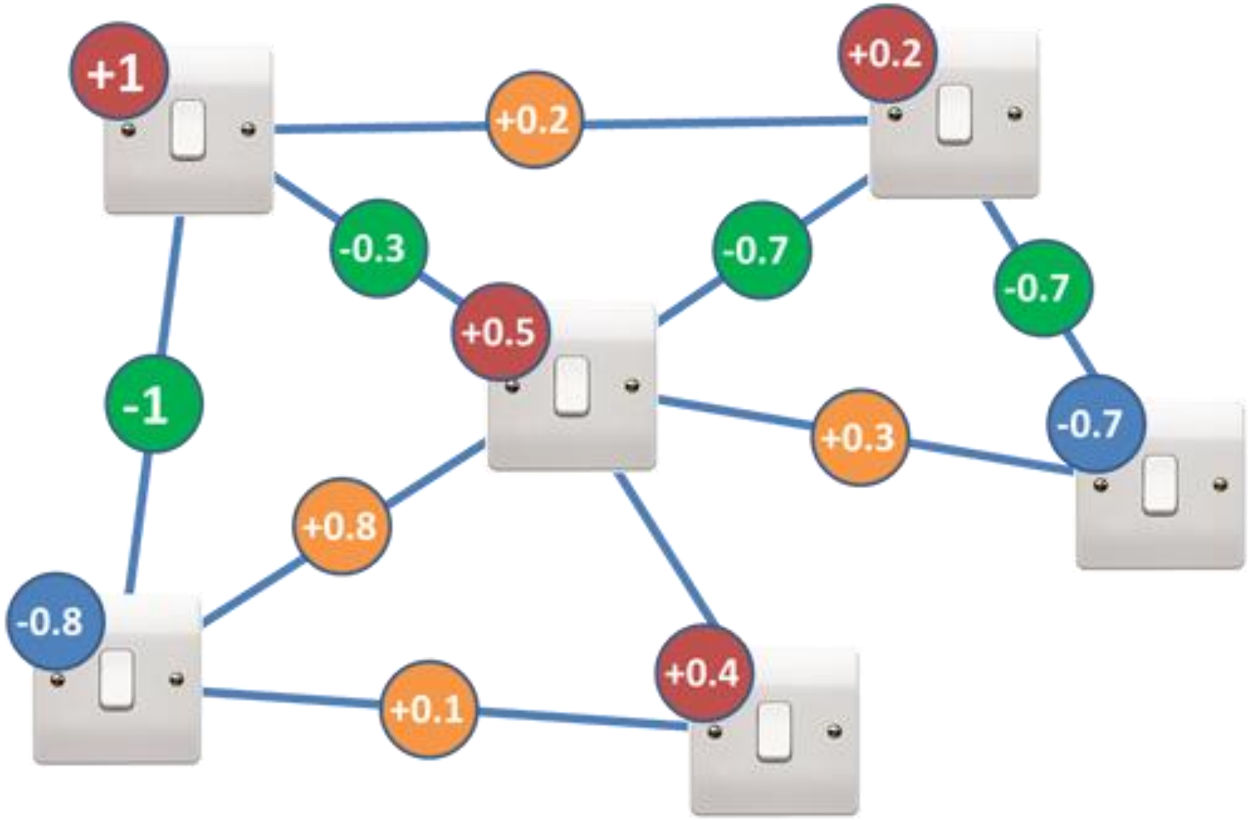


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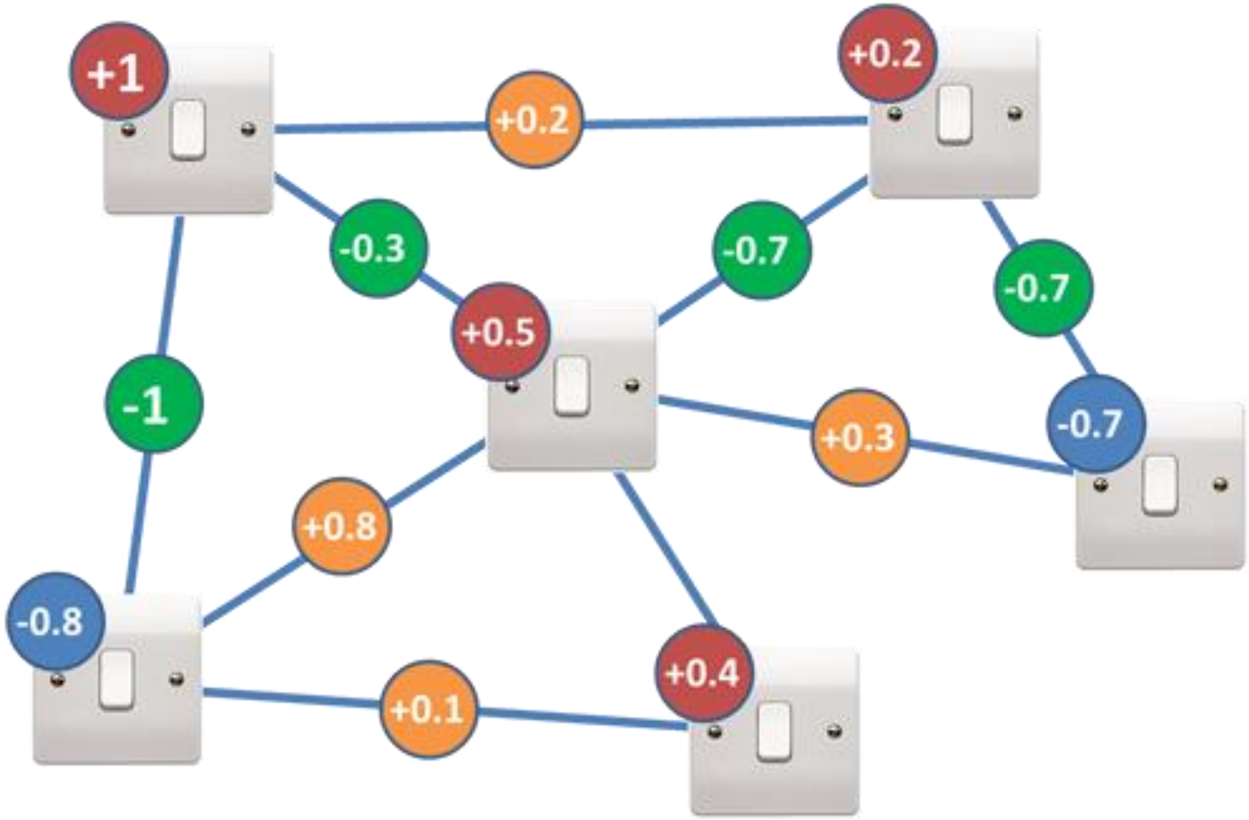


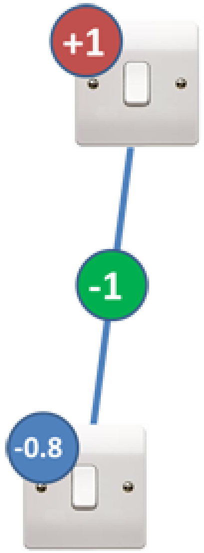


Adding another weight, J, which multiplies the product of the two switch settings.

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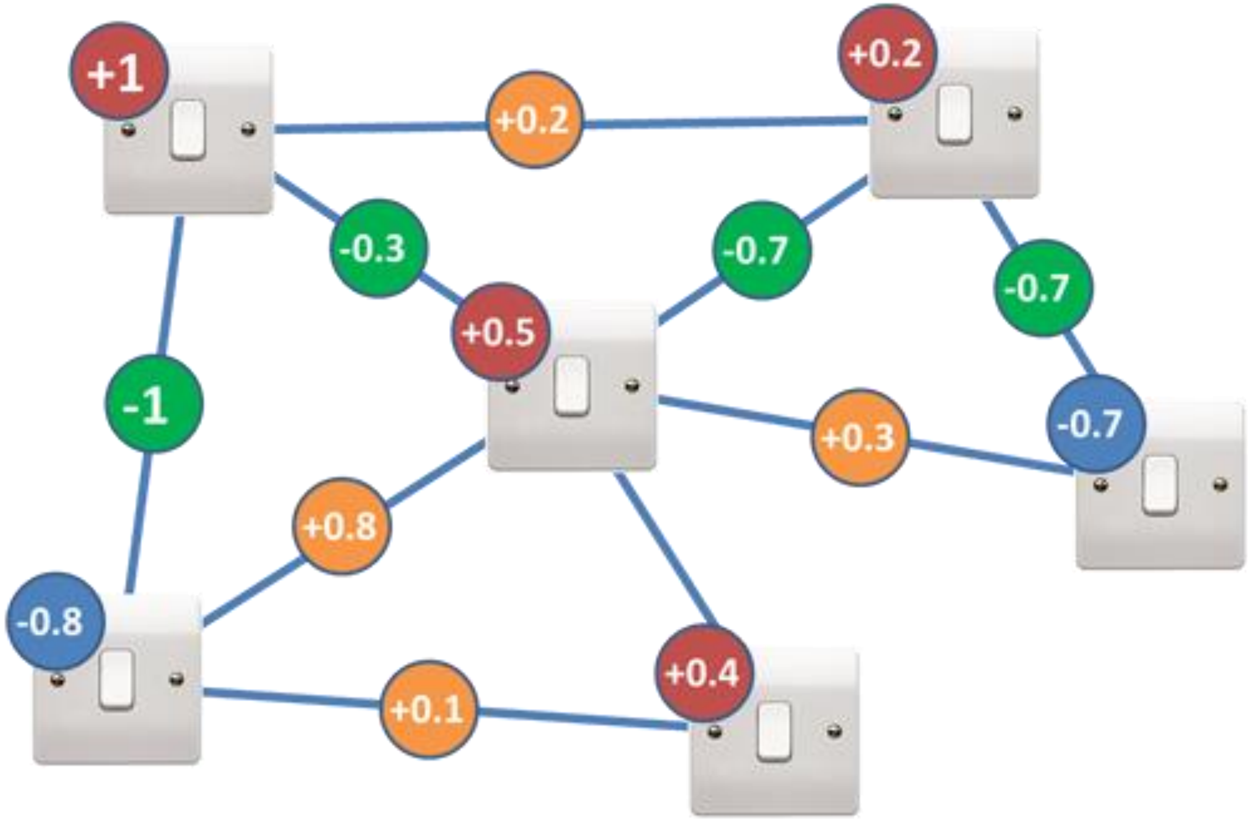


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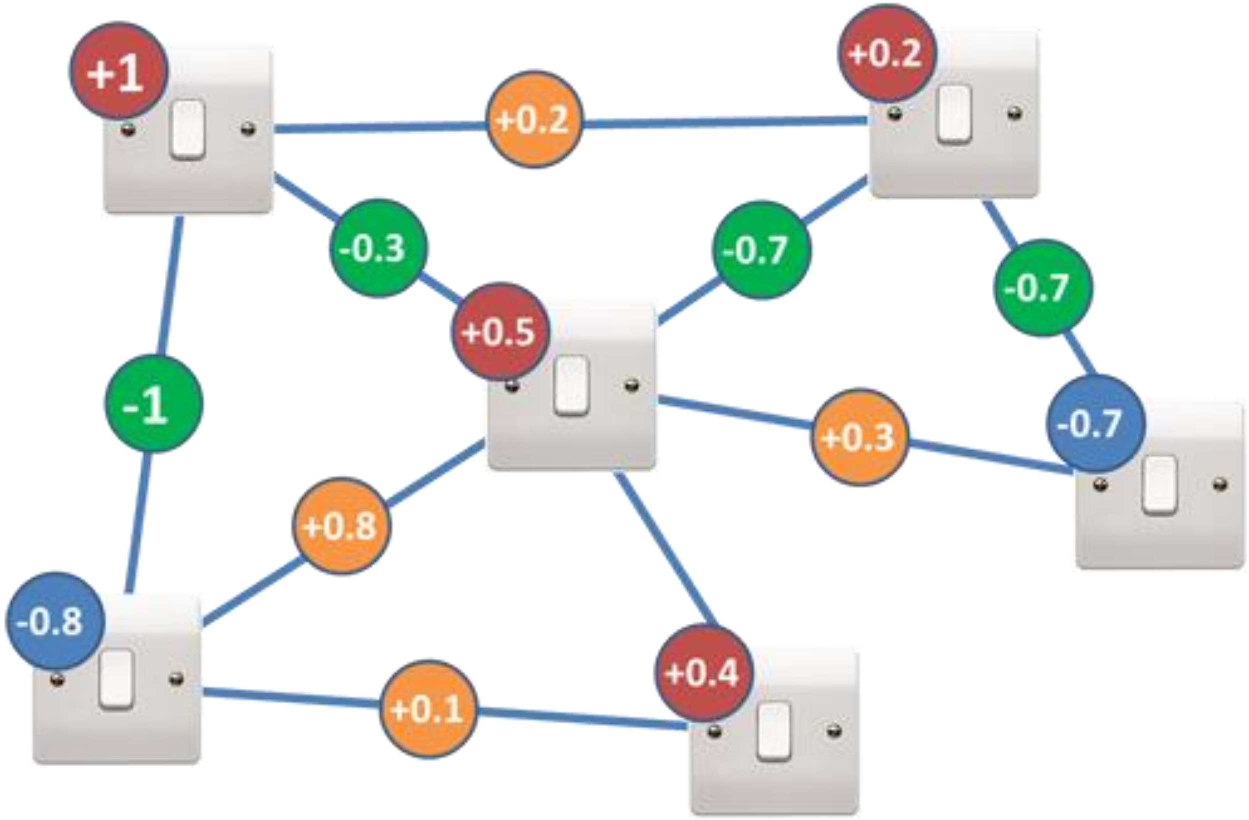


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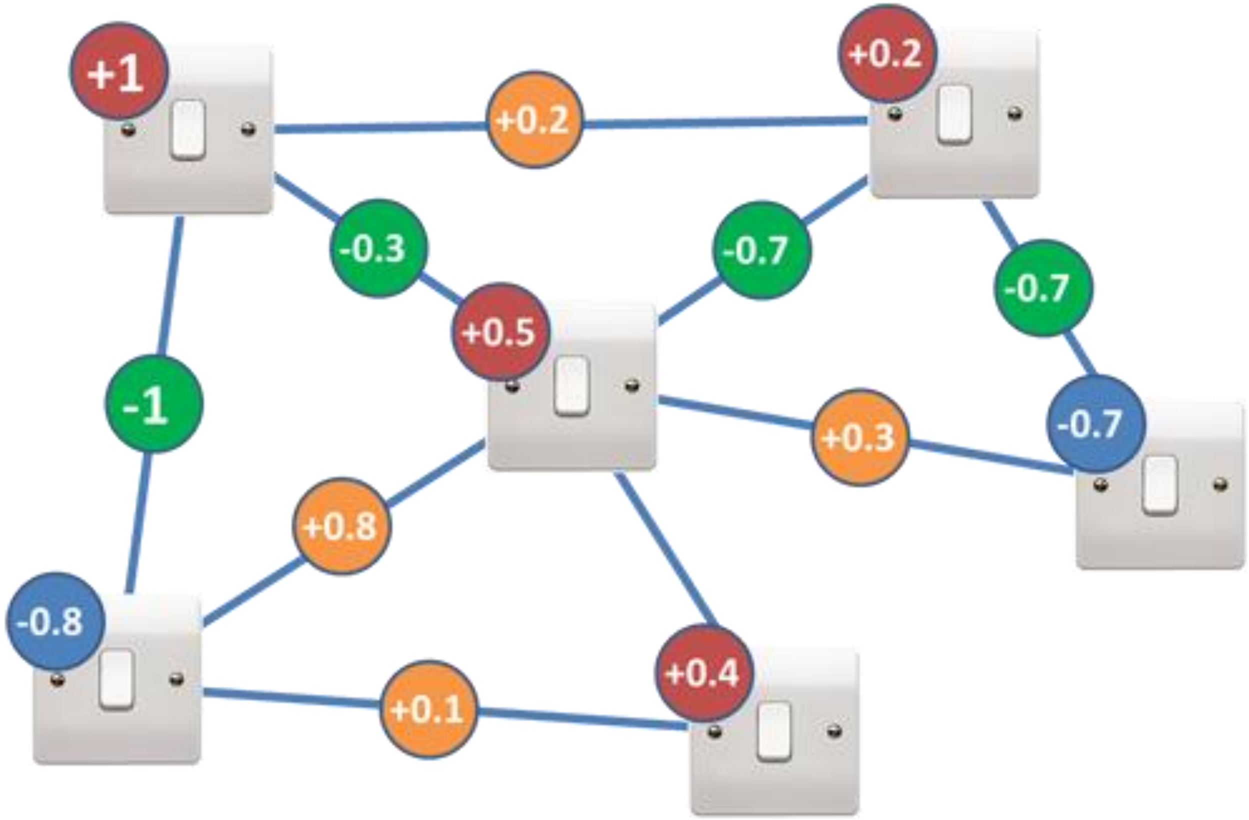
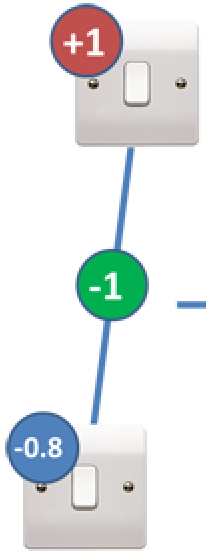
- We therefore add to the quantity to be minimized the contribution introduced by the couplers

# Game of Switches

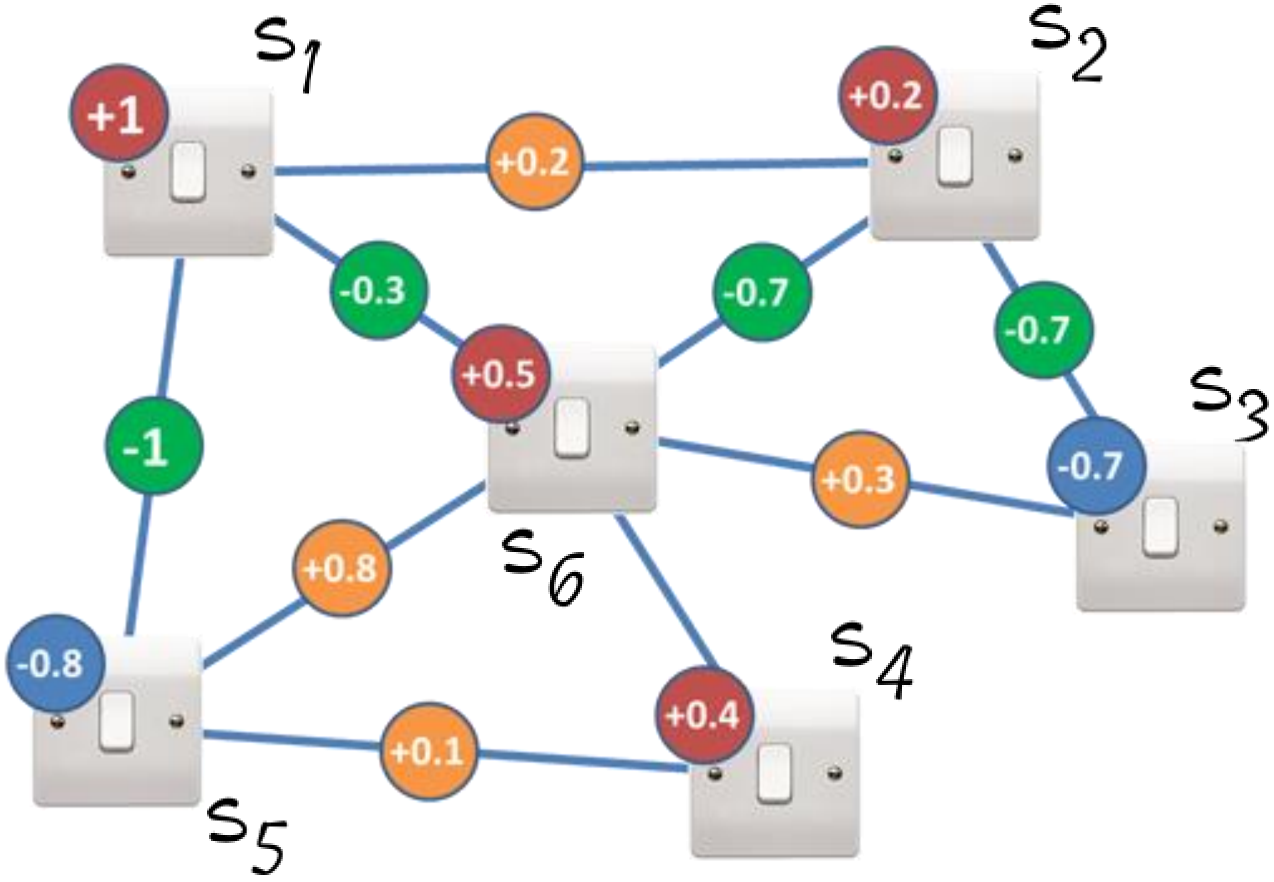



$$E(\mathbf{s}) = \sum_i h_i s_i + \boxed{\sum_{i,j} J_{i,j} s_i s_j}$$

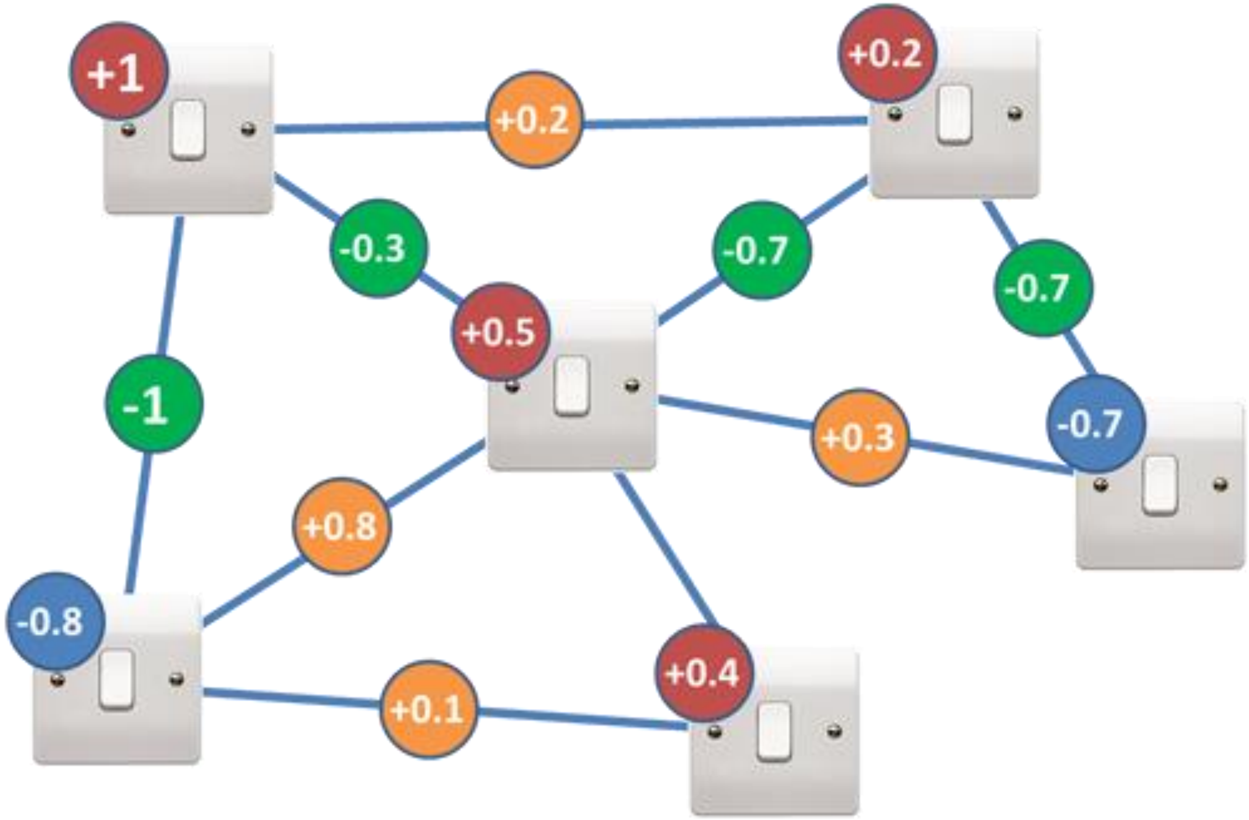
**Adding another weight, J, which multiplies the product of the two switch settings.**

# Game of Switches




$$s_1 + 0.2s_2 - 0.7s_3 + 0.4s_4 - 0.8s_5 + 0.5s_6 + 0.2s_1s_2 - 0.7s_2s_3 + 0.3s_3s_6 - 0.7s_2s_6 - 0.3s_1s_6 - s_1s_5 + 0.1s_5s_4 + s_6s_4$$

# Game of Switches




2 switches = $2^2$ =
4 possible answers

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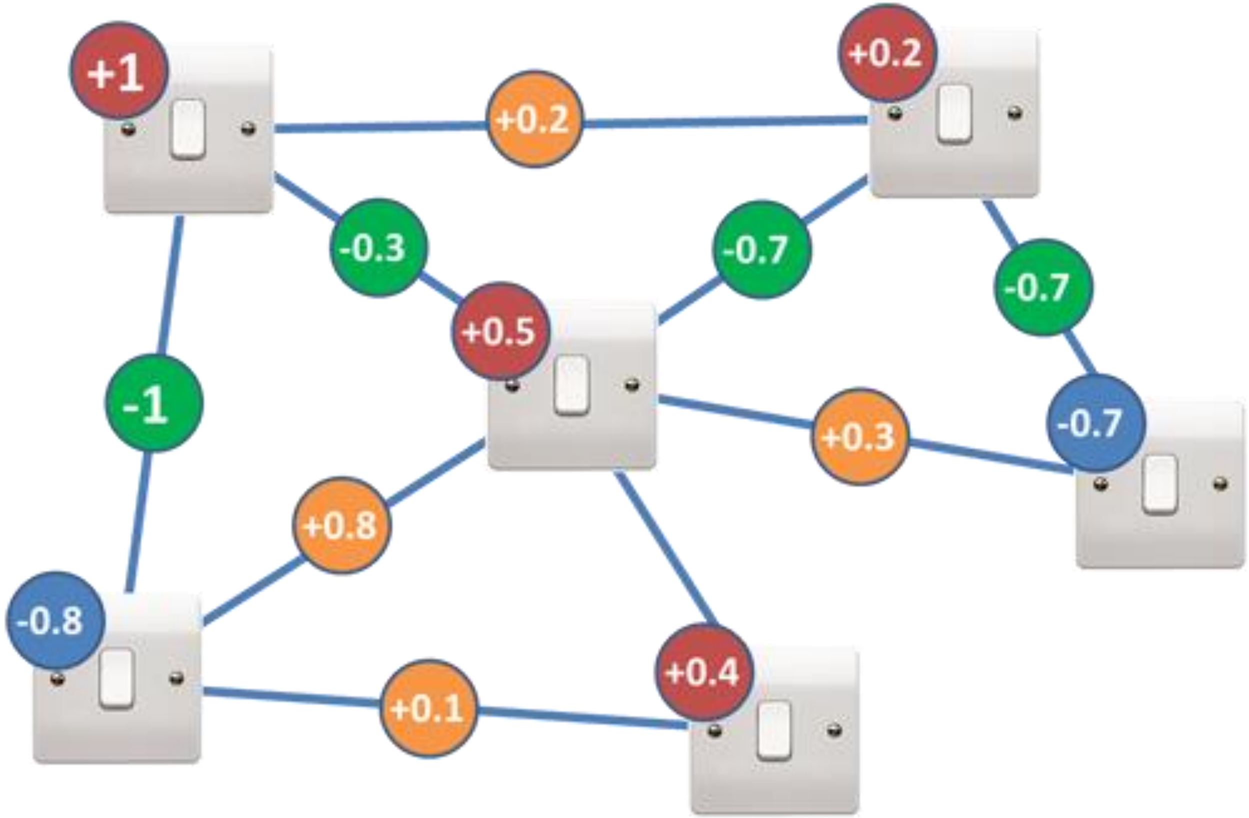


2 switches = $2^2$ =
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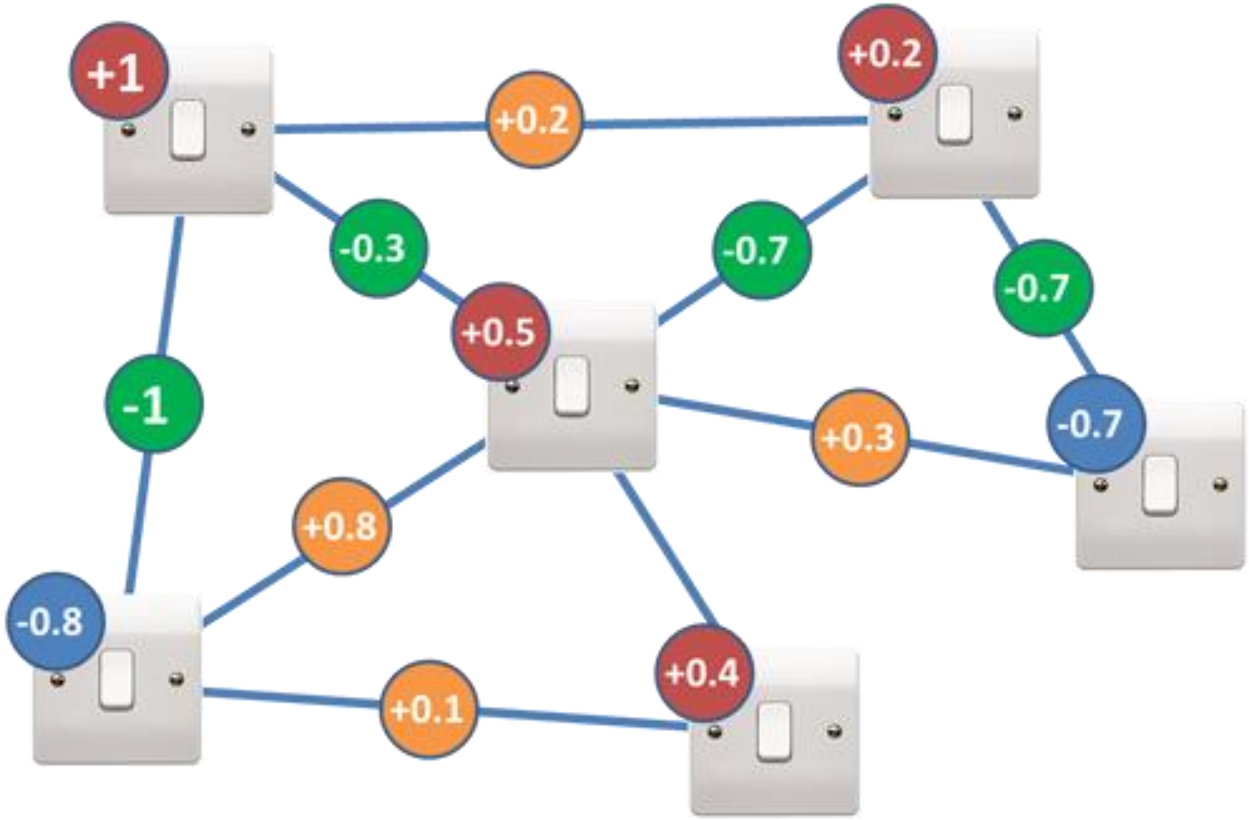
# Game of Switches

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- Where the diagonal terms of the matrix Q play the role of linear coefficients while the other non-zero elements are the quadratic coefficients. In matrix form

$$\min_{x \in \{0,1\}^n} x^T Q x.$$

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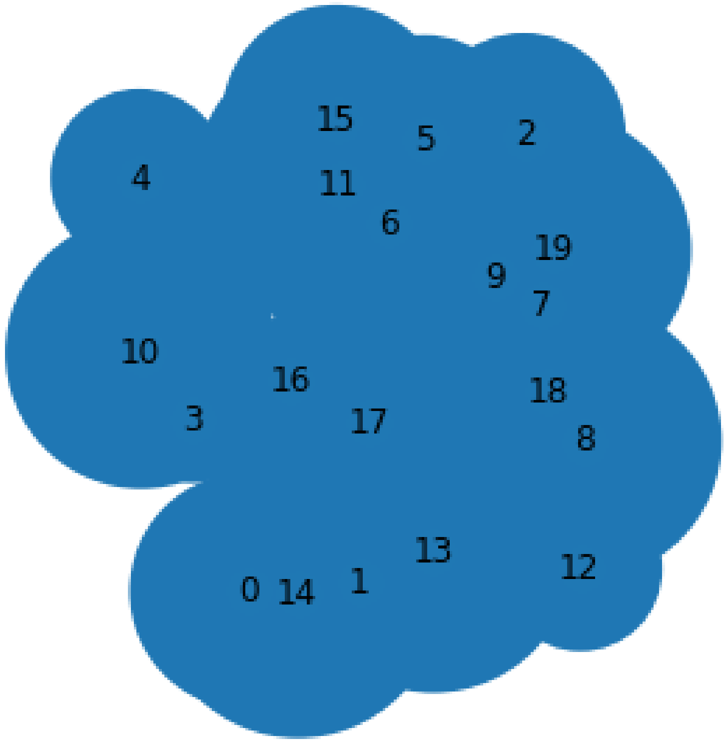

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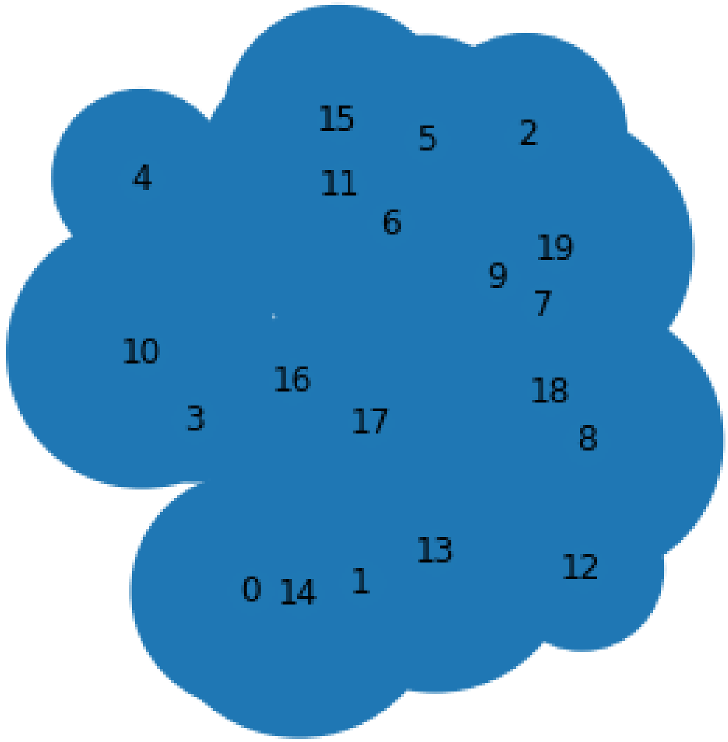
- Suppose we have a certain number of antennas and a certain number of possible sites to place these antennas.
- Each antenna with its signal can cover a certain area. When multiple signals overlap, however, unpleasant interference is generated
- **Our task is to position the antennas in order to maximize the surface covered by the signal and at the same time minimize interference between the antennas.**



# QUBO Problems

We define:

- The area covered by a single antenna such as the area of the circle whose radius is the parameter that describes the range of action of each individual antenna (problem data)

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- where $r_i$ and $r_j$ are the parameters relating to the range of action of the antennas i and j and $dist(c_i, c_j)$ is the distance between the points where the antennas are positioned

# QUBO Problems

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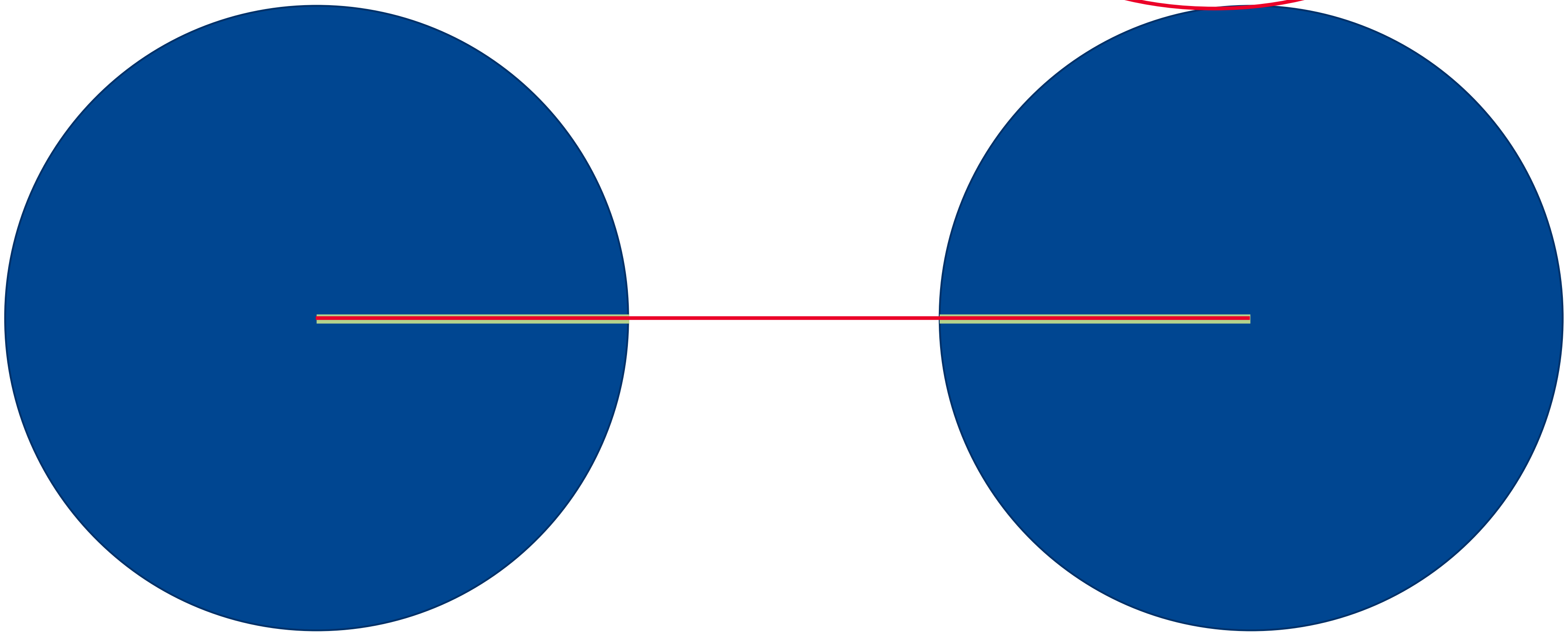

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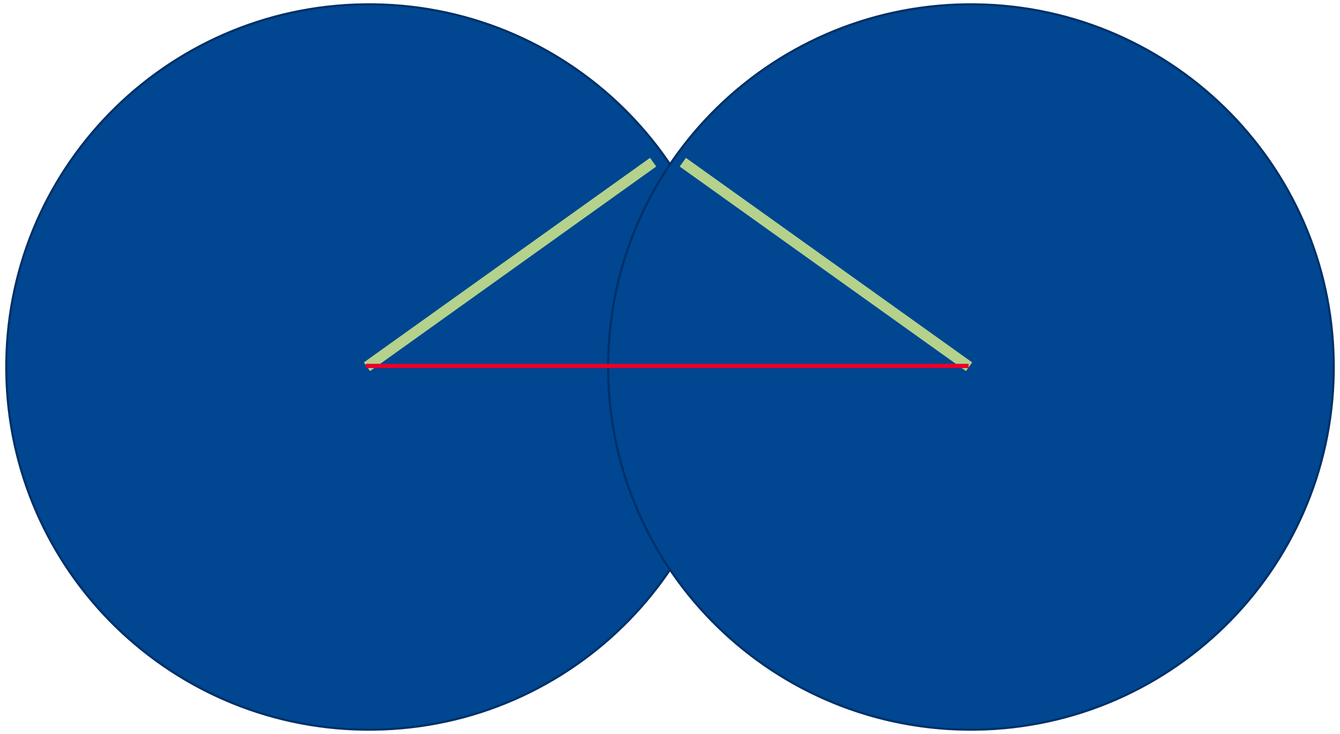

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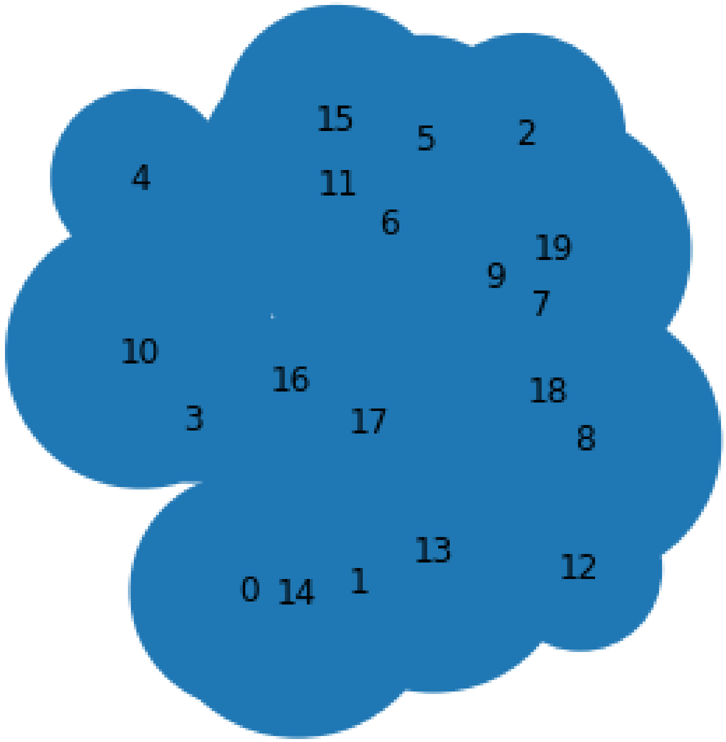

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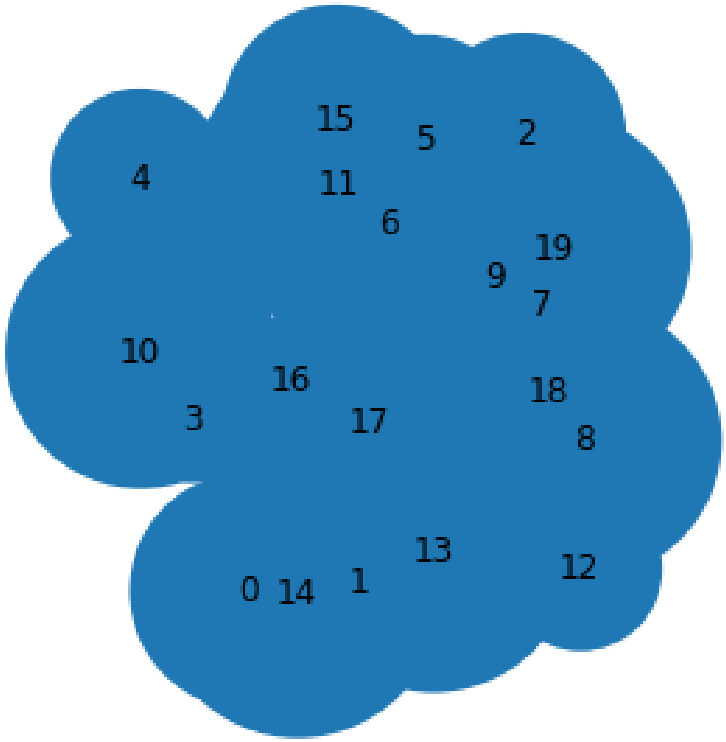
$$[q_0, \dots, q_{19}]$$



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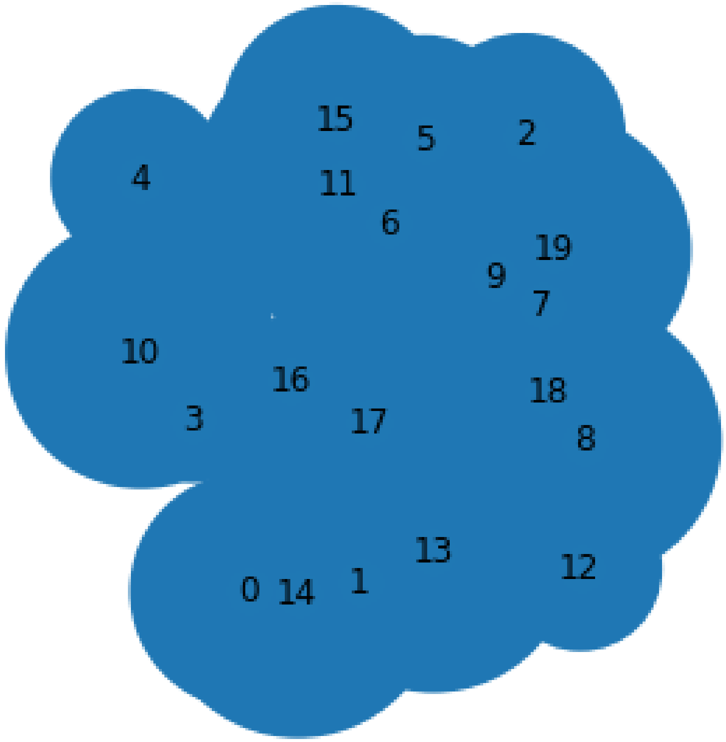

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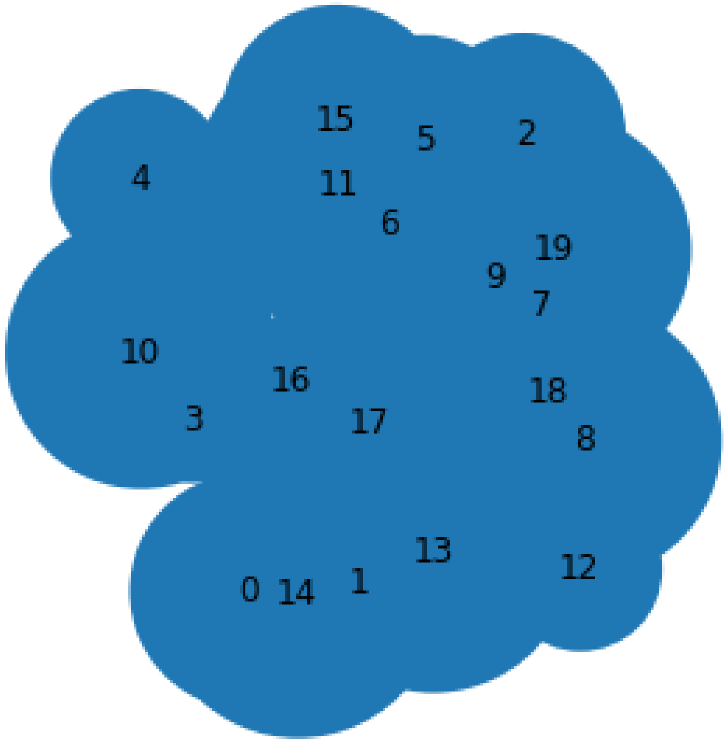

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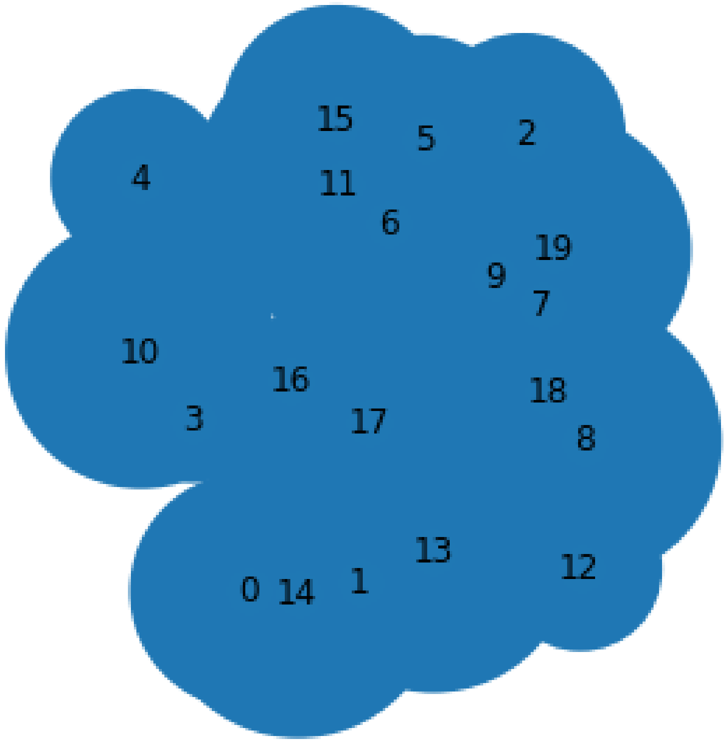


$$QUBO =$$

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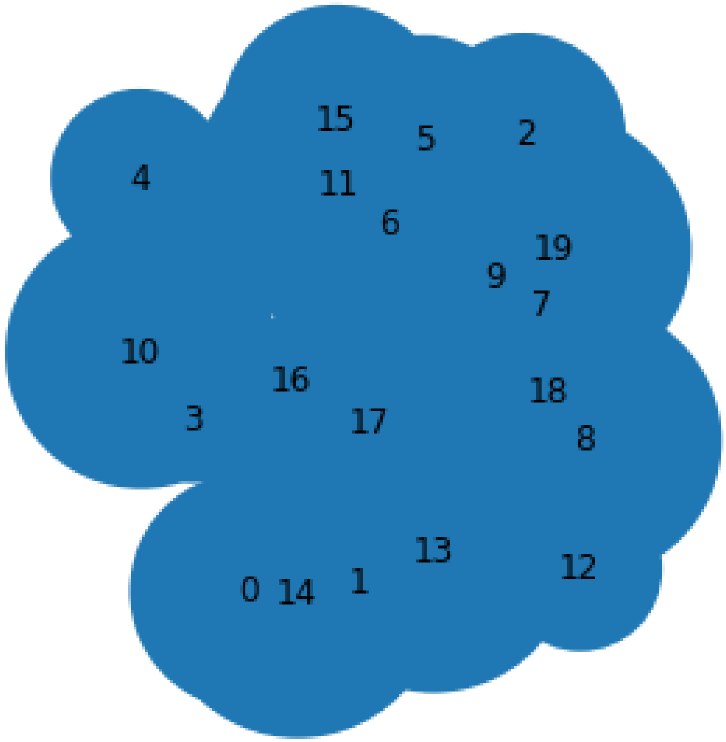
$$A_i = r_i^2 \cdot \pi \qquad\qquad B_{ij} = \rho_{ij}^2 \cdot \pi$$

- Minimize interference

$$QUBO = \sum_{i<j} B_{ij} q_i q_j$$



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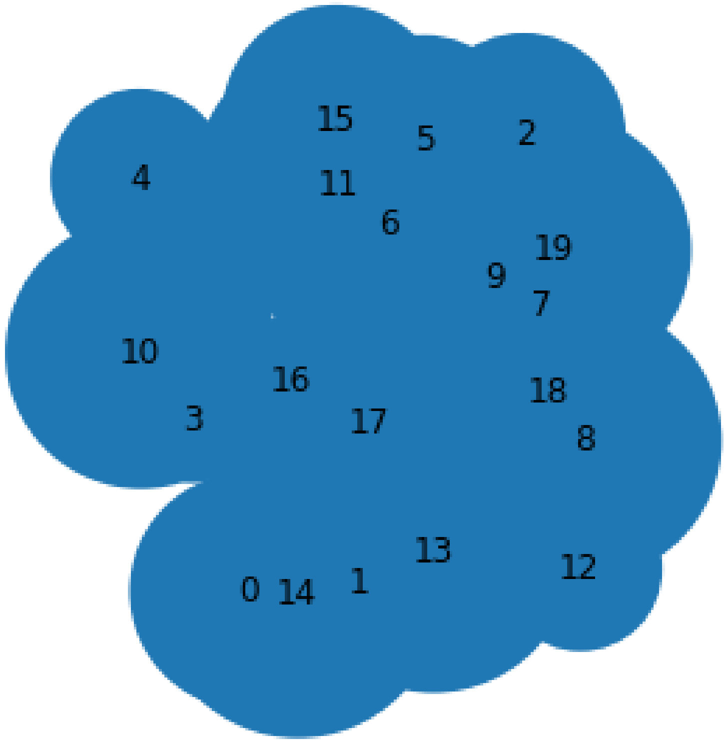

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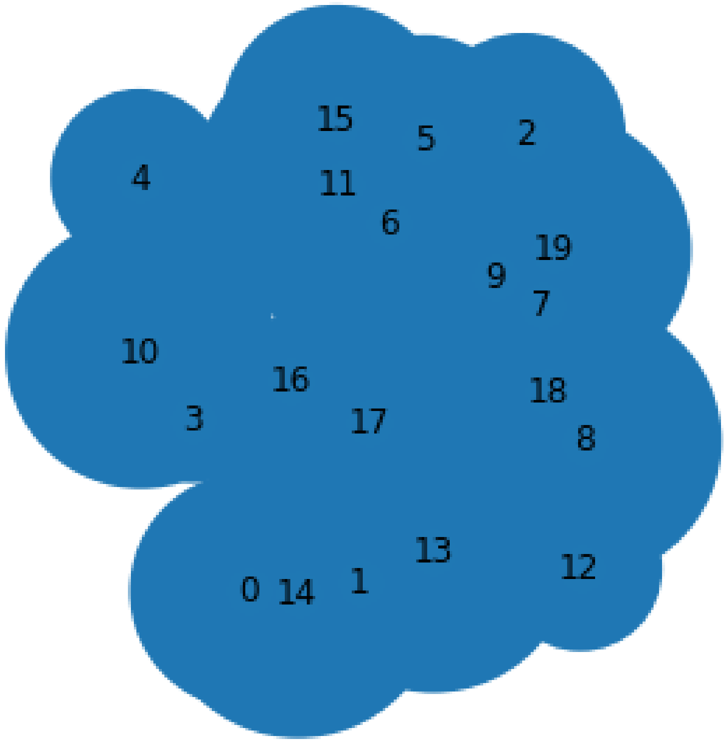

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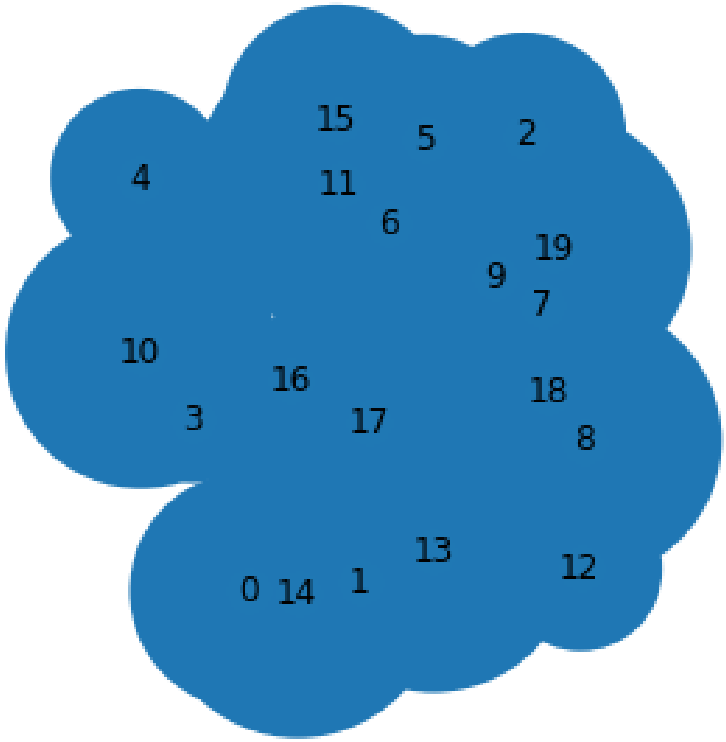

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$$x_i \mapsto \frac{s_i + 1}{2}$$

$$E_{ising}(s) = \sum_{i=1}^{N} h_i s_i + \sum_{i=1}^{N} \sum_{j=i+1}^{N} J_{i,j} s_i s_j$$

$$E_{QUBO}(x) = \sum_{i} Q_{i,i} x_i + \sum_{i<j} Q_{i,j} x_i x_j$$

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- More generally, any mathematical problem can be mapped into a QUBO problem
- You just have to understand if it's worth it :)

# Programming a Quantum Annealer

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- Variables: Type Binary (0/1)

```
>>> from pyqubo import Binary
>>> x1, x2 = Binary('x1'), Binary('x2')
>>> H = 2*x1*x2 + 3*x1
>>> pprint(H.compile().to_qubo()) # doctest: +SKIP
({('x1', 'x1'): 3.0, ('x1', 'x2'): 2.0, ('x2', 'x2'): 0.0}, 0.0)
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- Variables: Type Spin (+1/-1)

```
>>> from pyqubo import Spin
>>> s1, s2 = Spin('s1'), Spin('s2')
>>> H = 2*s1*s2 + 3*s1
>>> pprint(H.compile().to_qubo()) # doctest: +SKIP
({('s1', 's1'): 2.0, ('s1', 's2'): 8.0, ('s2', 's2'): -4.0}, -1.0)
```

# Programming a Quantum Annealer

- Arrays of Binary type variables (same for Spin type variables)

```
>>> from pyqubo import Array
>>> x = Array.create('x', shape=(2, 3), vartype='BINARY')
>>> x[0, 1] + x[1, 2]
(Binary(x[0][1])+Binary(x[1][2]))
```

# Programming a Quantum Annealer

- Arrays of Binary type variables (same for Spin type variables)

```
>>> from pyqubo import Array
>>> numbers = [4, 2, 7, 1]
>>> s = Array.create('s', shape=4, vartype='SPIN')
>>> H = sum(n * s for s, n in zip(s, numbers))**2
>>> model = H.compile()
>>> qubo, offset = model.to_qubo()
>>> pprint(qubo) # doctest: +SKIP
{('s[0]', 's[0]'): -160.0,
 ('s[0]', 's[1]'): 64.0,
 ('s[0]', 's[2]'): 224.0,
 ('s[0]', 's[3]'): 32.0,
 ('s[1]', 's[1]'): -96.0,
 ('s[1]', 's[2]'): 112.0,
 ('s[1]', 's[3]'): 16.0,
 ('s[2]', 's[2]'): -196.0,
 ('s[2]', 's[3]'): 56.0,
 ('s[3]', 's[3]'): -52.0}
```

# Programming a Quantum Annealer

- Construct a QUBO problem with PyQUBO

```
>>> from pyqubo import Binary
>>> a, b = Binary('a'), Binary('b')
>>> M = 5.0
>>> H = 2*a + b + M*(a+b-1)**2
>>> model = H.compile()
>>> qubo, offset = model.to_qubo() # QUBO with M=5.0
>>> M = 6.0
>>> H = 2*a + b + M*(a+b-1)**2
>>> model = H.compile()
>>> qubo, offset = model.to_qubo() # QUBO with M=6.0
```

# Programming a Quantum Annealer

- Construct a QUBO problem with PyQUBO (with Placeholders)

```
>>> from pyqubo import Binary
>>> a, b = Binary('a'), Binary('b')
>>> M = 5.0
>>> H = 2*a + b + M*(a+b-1)**2
>>> model = H.compile()
>>> qubo, offset = model.to_qubo() # QUBO with M=5.0
>>> M = 6.0
>>> H = 2*a + b + M*(a+b-1)**2
>>> model = H.compile()
>>> qubo, offset = model.to_qubo() # QUBO with M=6.0
```

```
>>> from pyqubo import Placeholder
>>> a, b = Binary('a'), Binary('b')
>>> M = Placeholder('M')
>>> H = 2*a + b + M*(a+b-1)**2
>>> model = H.compile()
>>> qubo, offset = model.to_qubo(feed_dict={'M': 5.0})
```

# Programming a Quantum Annealer

- Solve a problem set via pyQUBO
- After setting the Hamiltonian of the problem, it must be compiled and transformed into a bqm object

```
>>> from pyqubo import Binary
>>> x1, x2 = Binary('x1'), Binary('x2')
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>>> model = H.compile()
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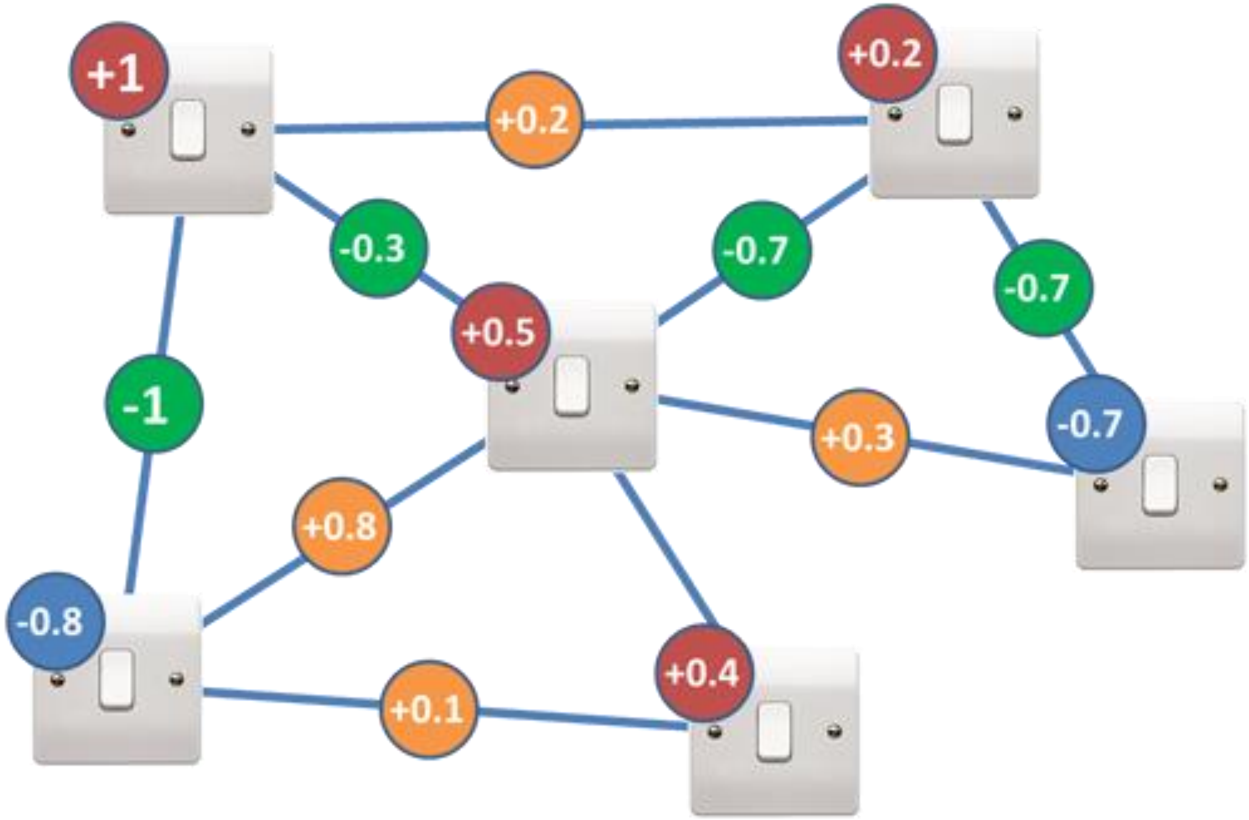
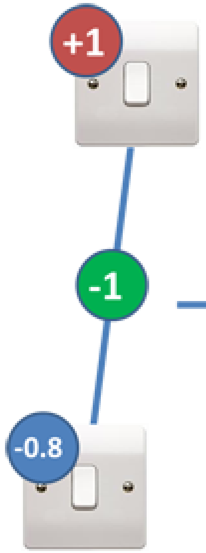
```

```
>>> import neal
>>> sa = neal.SimulatedAnnealingSampler()
>>> sampleset = sa.sample(bqm, num_reads=10)
>>> decoded_samples = model.decode_sampleset(sampleset)
>>> best_sample = min(decoded_samples, key=lambda x: x.energy)
>>> pprint(best_sample.sample)
{'x1': 0, 'x2': 1}
```

# Exercise 1: Game of Switches

- Try to implement the Game of Switches




$$E(\mathbf{s}) = \sum_i h_i s_i + \boxed{\sum_{i,j} J_{i,j} s_i s_j}$$

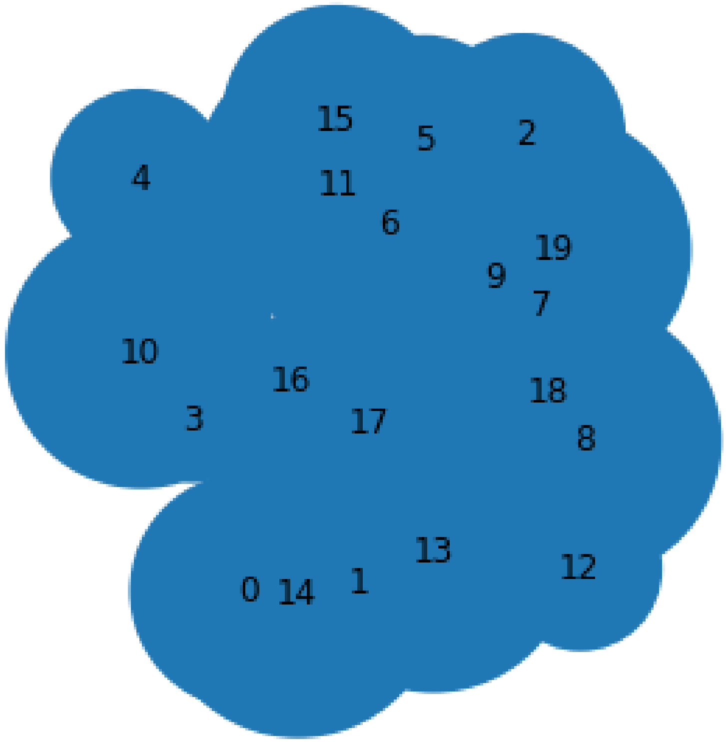
**Adding another weight, J, which multiplies the product of the two switch settings.**

# Exercise 2: Antenna Placement

- Try to implement the Antenna Placement Problem

$$A_i = r_i^2 \cdot \pi \qquad B_{ij} = \rho_{ij}^2 \cdot \pi$$

$$QUBO = -\sum_{i=0}^{N} A_i q_i + \alpha \sum_{i<j} B_{ij} q_i q_j$$



# Add a Constraint to a QUBO Problem

- By definition, a QUBO problem admits no constraints

**Q**uadratic
**U**nconstrained
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- Still, there is a way.

# Add a Constraint to a QUBO Problem

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- Suppose we want to add the following constraint to our antenna optimization problem

- *Let F be the exact number of antennas to be placed*

- Remembering the mathematical formulation of our problem, requested constraint can be seen as

$$\sum_{i=0}^{N} q_i = F$$

# Add a Constraint to a QUBO Problem

- Let's do some math

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$$\sum_{i=0}^{N} q_i = F \quad \Rightarrow \quad \min\left(\sum_{i=0}^{N} q_i - F\right)^2$$

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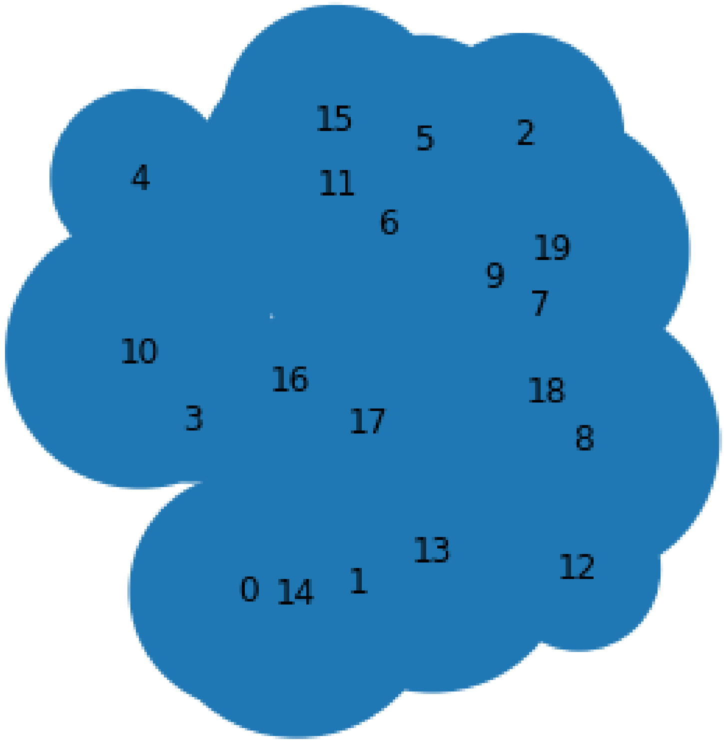

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- Consequently, penalty function can be seen as

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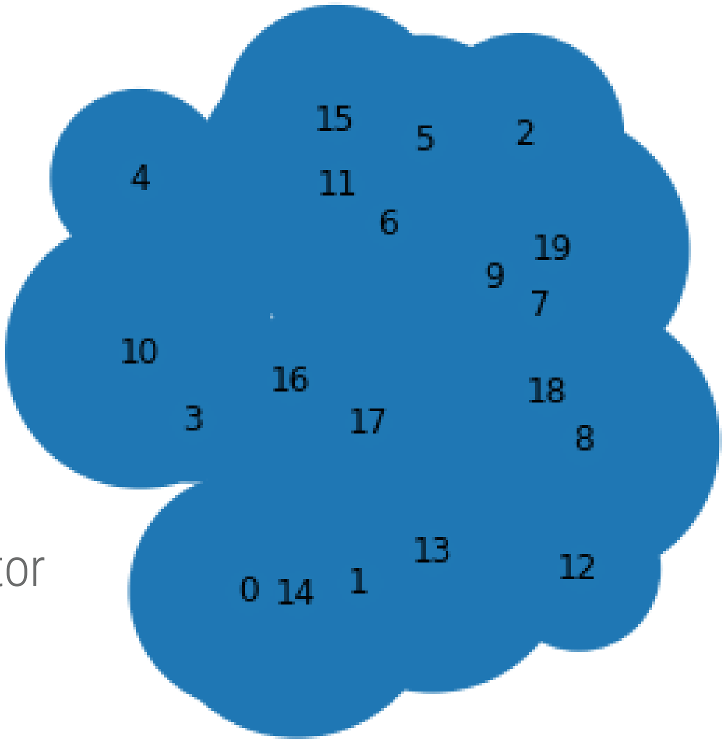


- Implement and configure L vector
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$$\sum_{i=0}^{N} q_i \leq F$$

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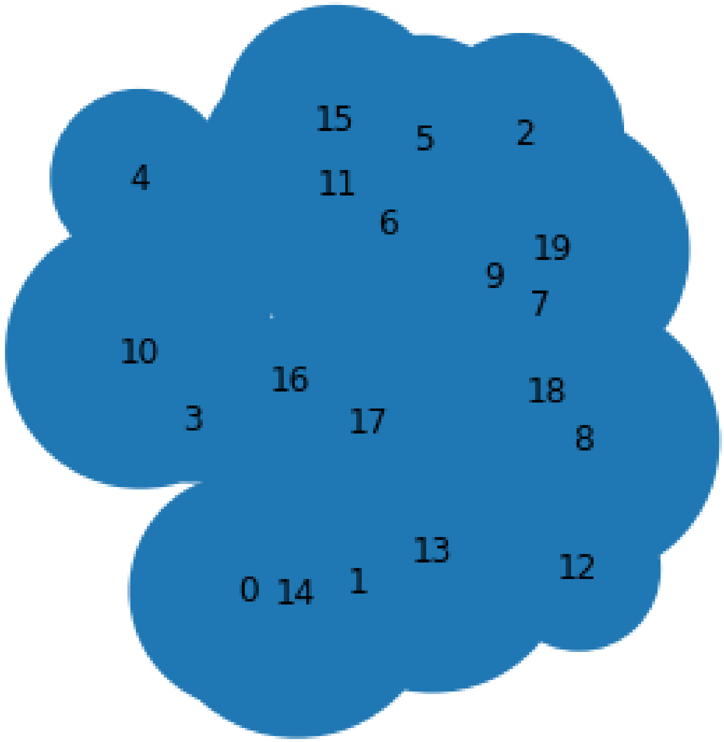

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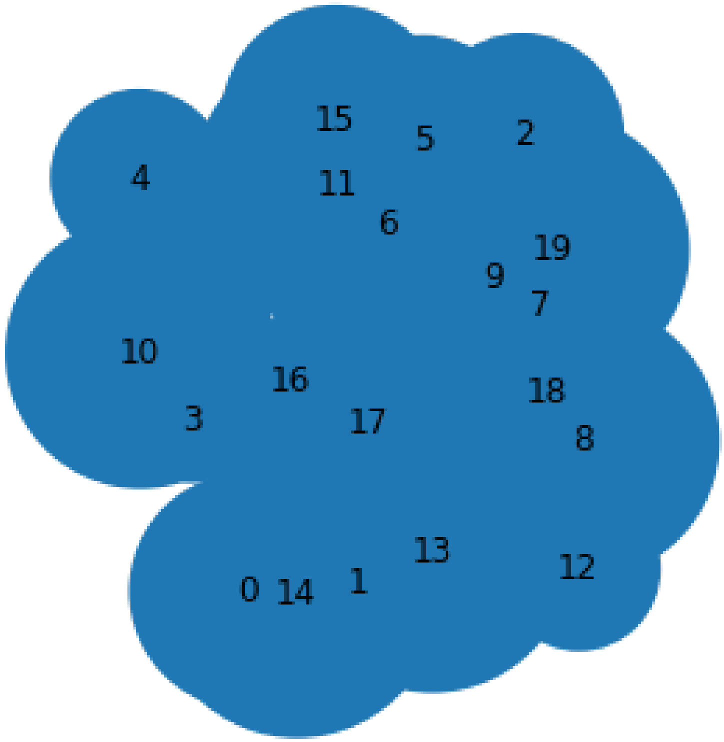

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| $\mathbf{x,y,z}$ | $\mathbf{xyz}$ | $\mathbf{x+y+z-2}$ | $\max_{\mathbf{w}} \{\mathbf{w(x+y+z-2)}\}$ |
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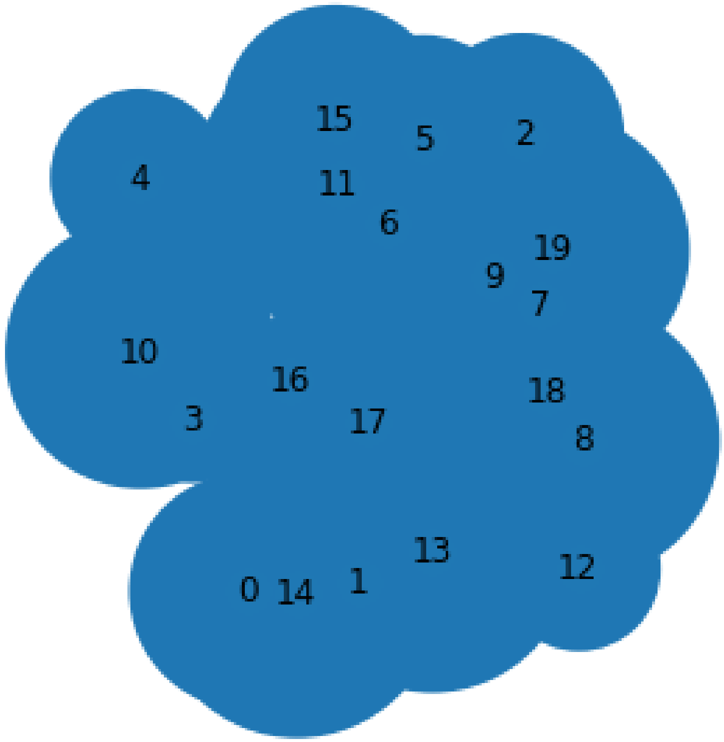
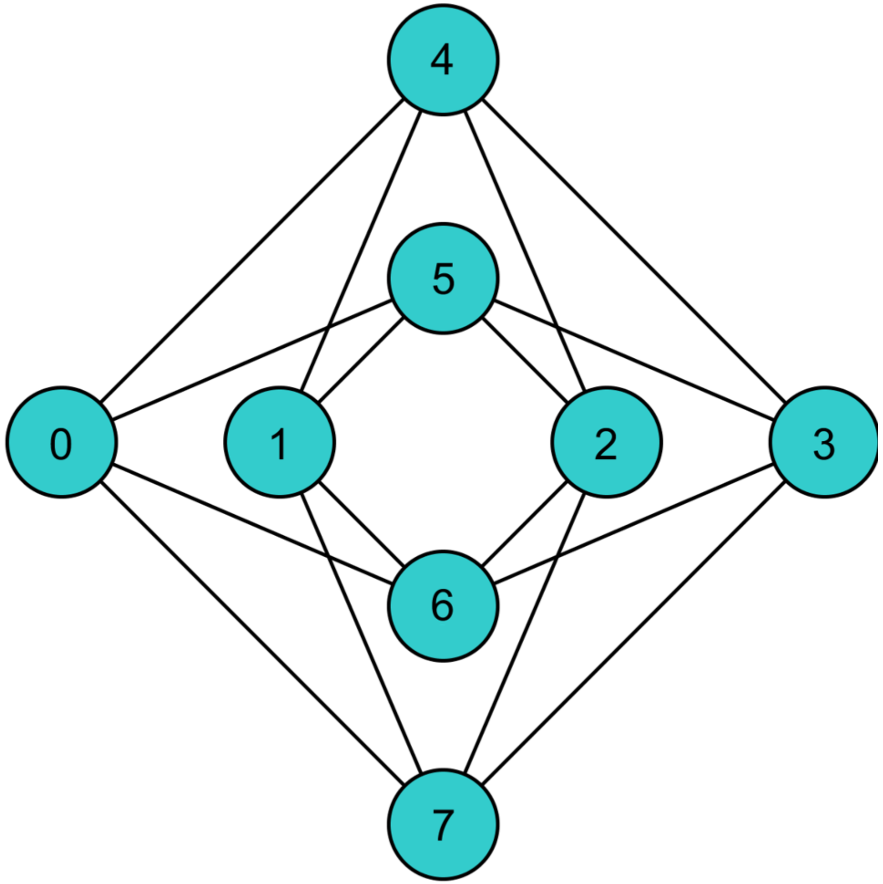



- Add High Order Terms to QUBO problem with pyqubo

# Graphs

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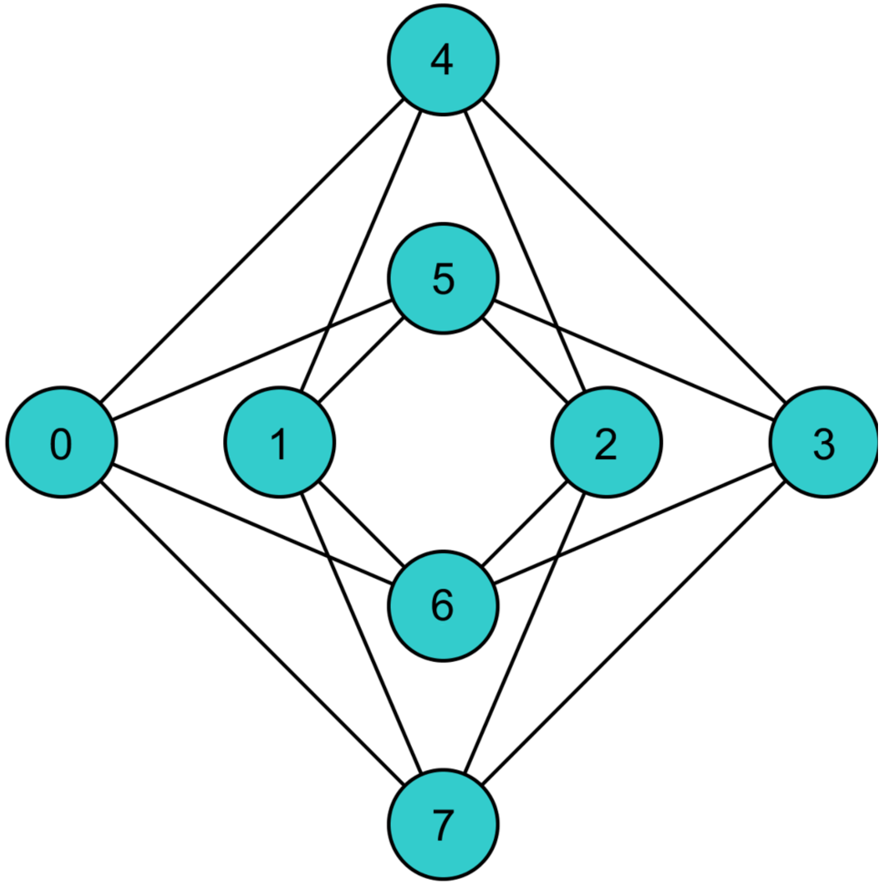


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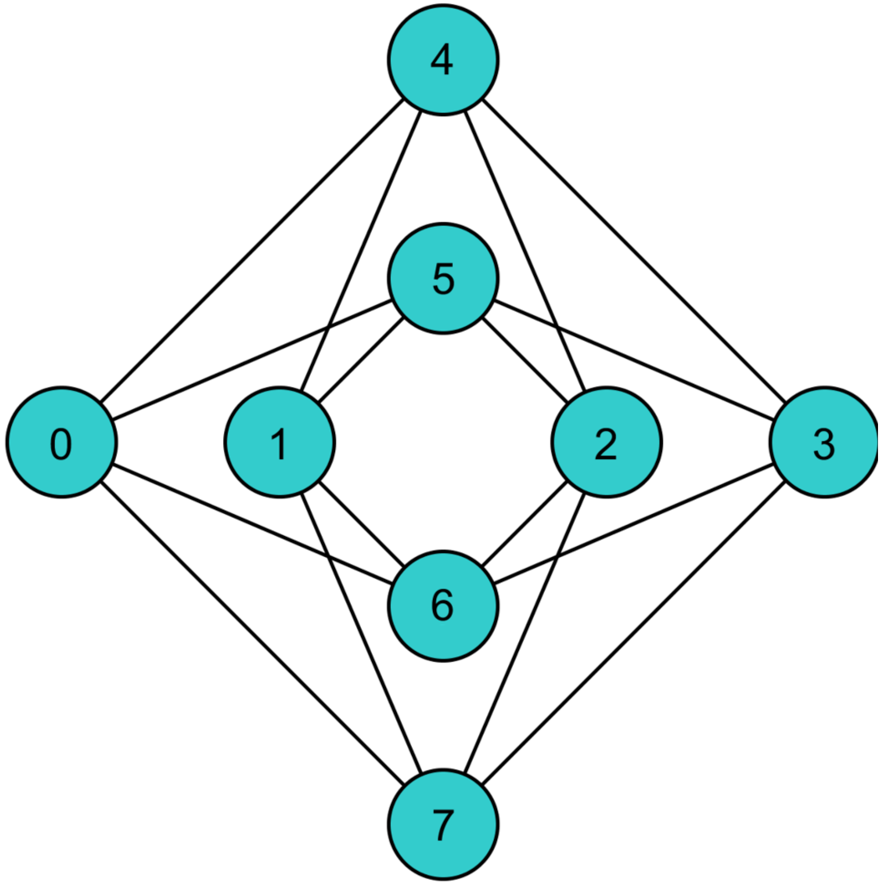


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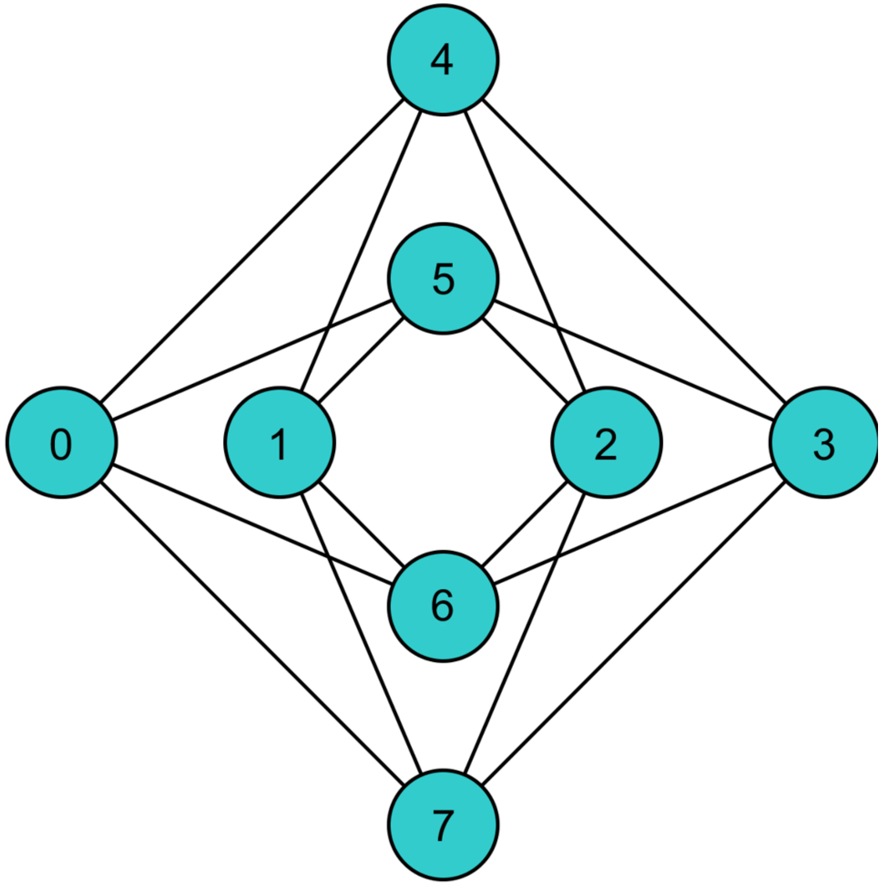


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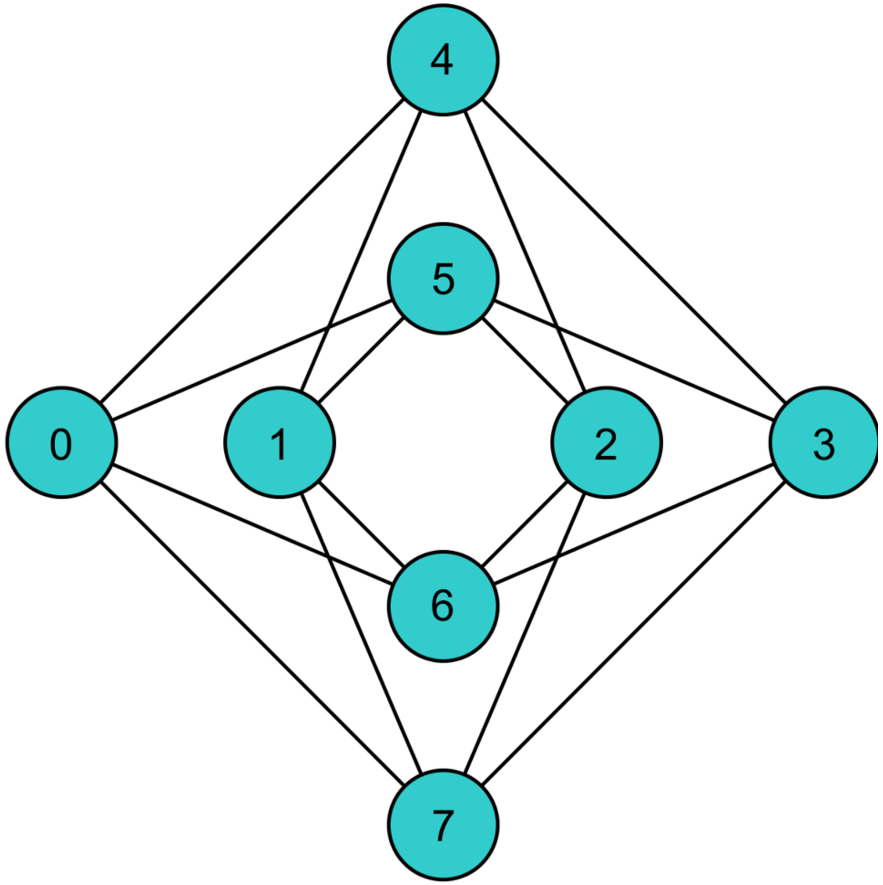


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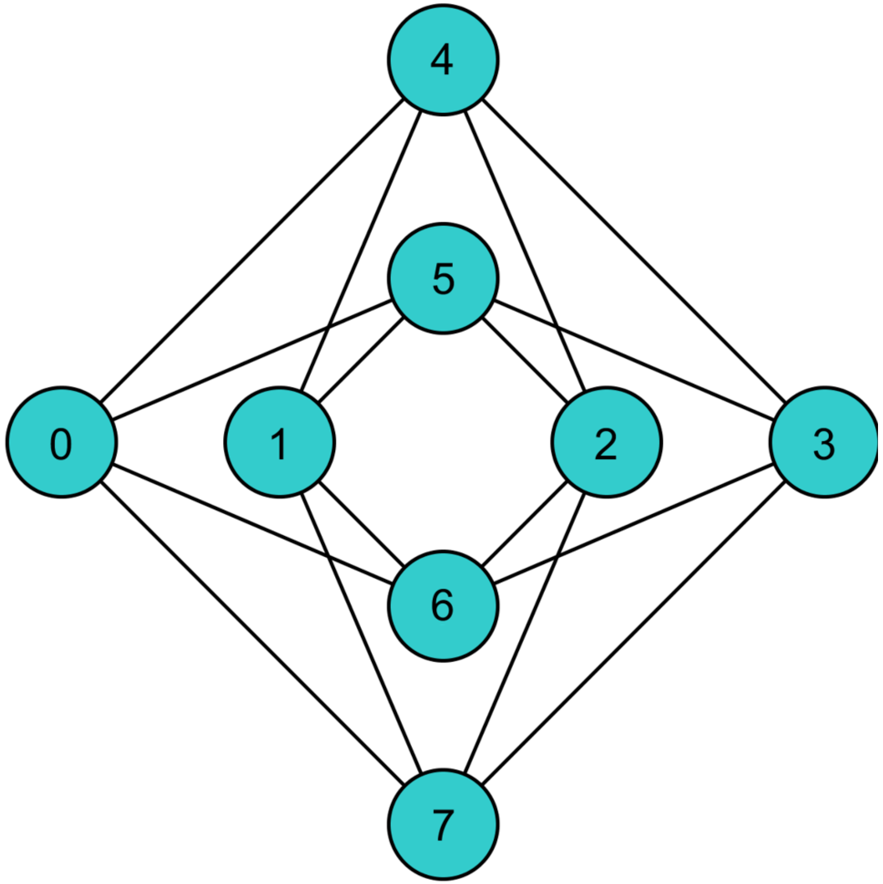


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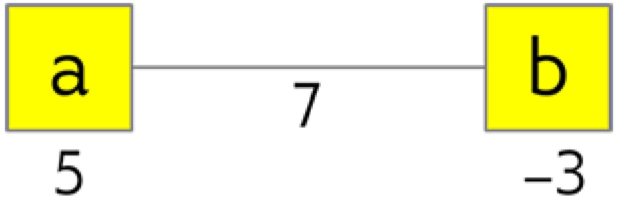
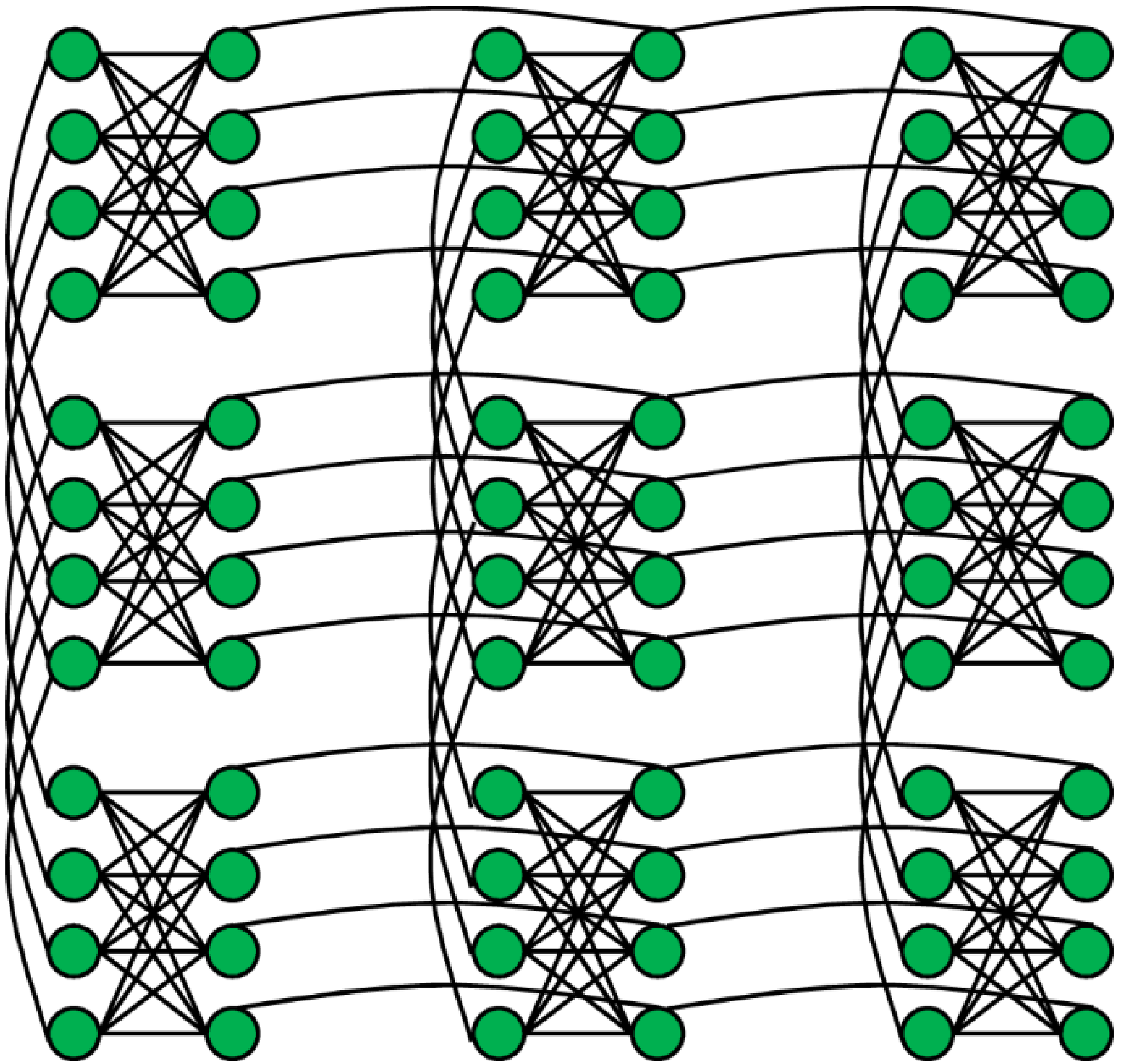


# Embedding a problem on a graph



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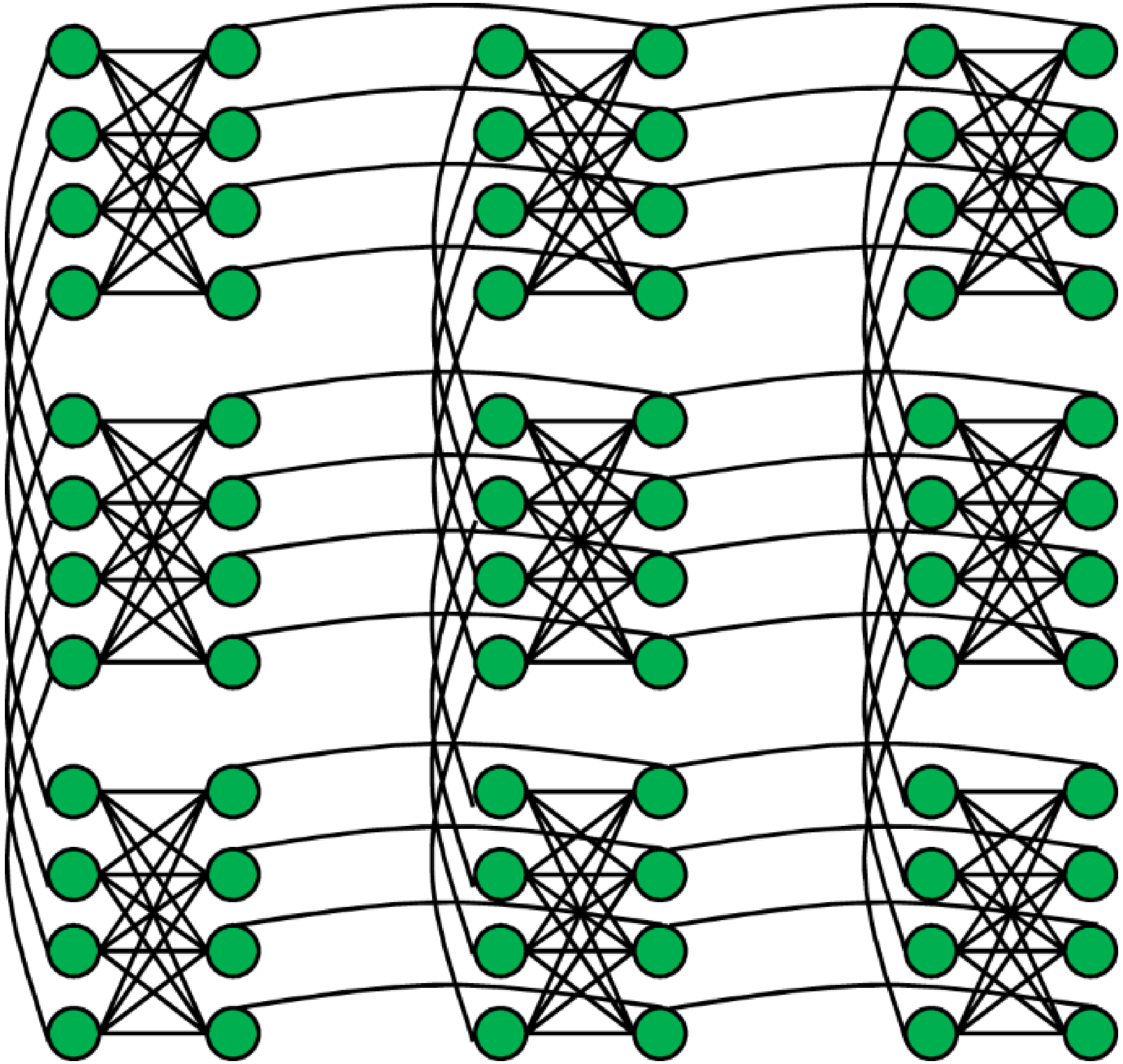

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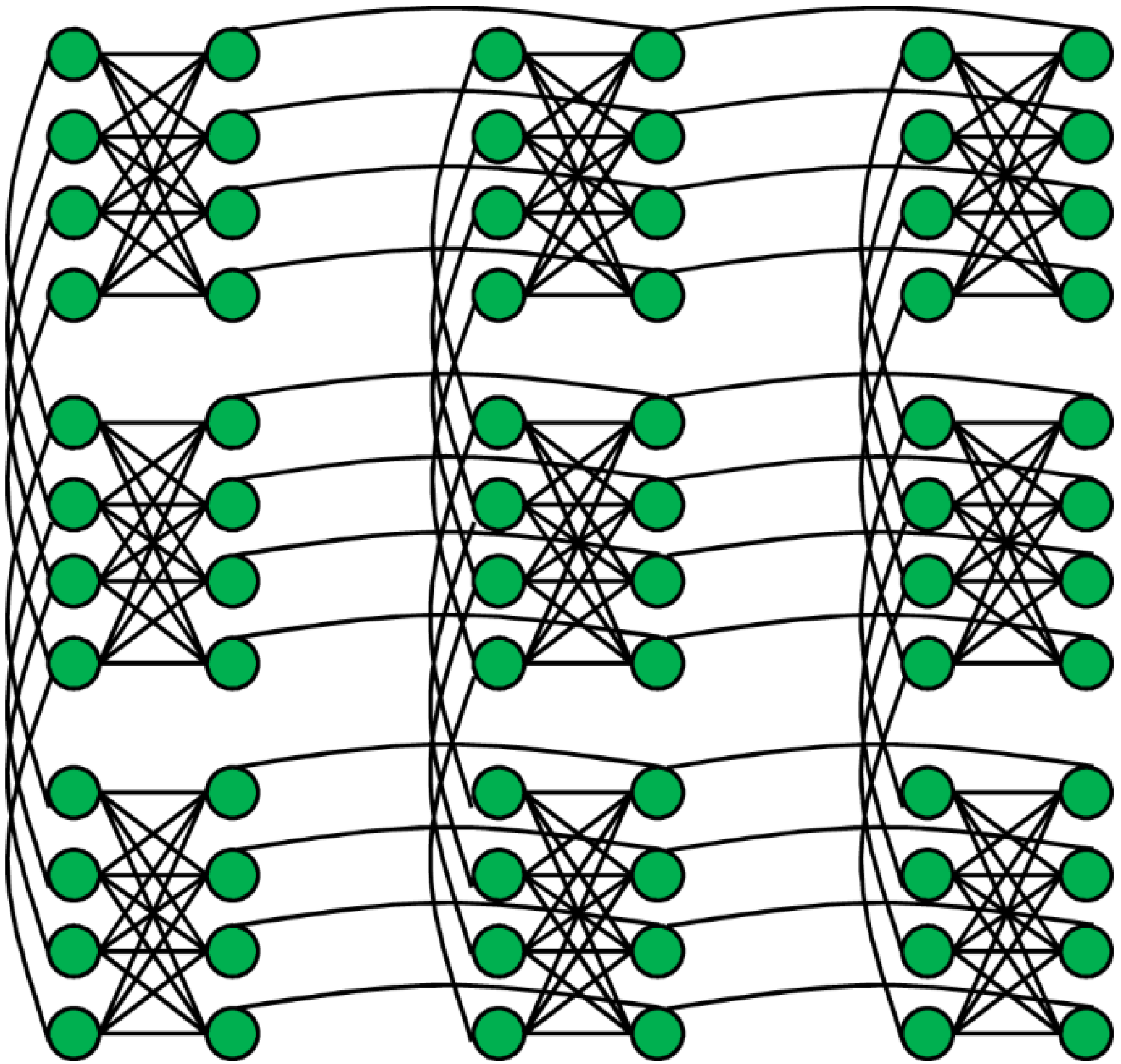

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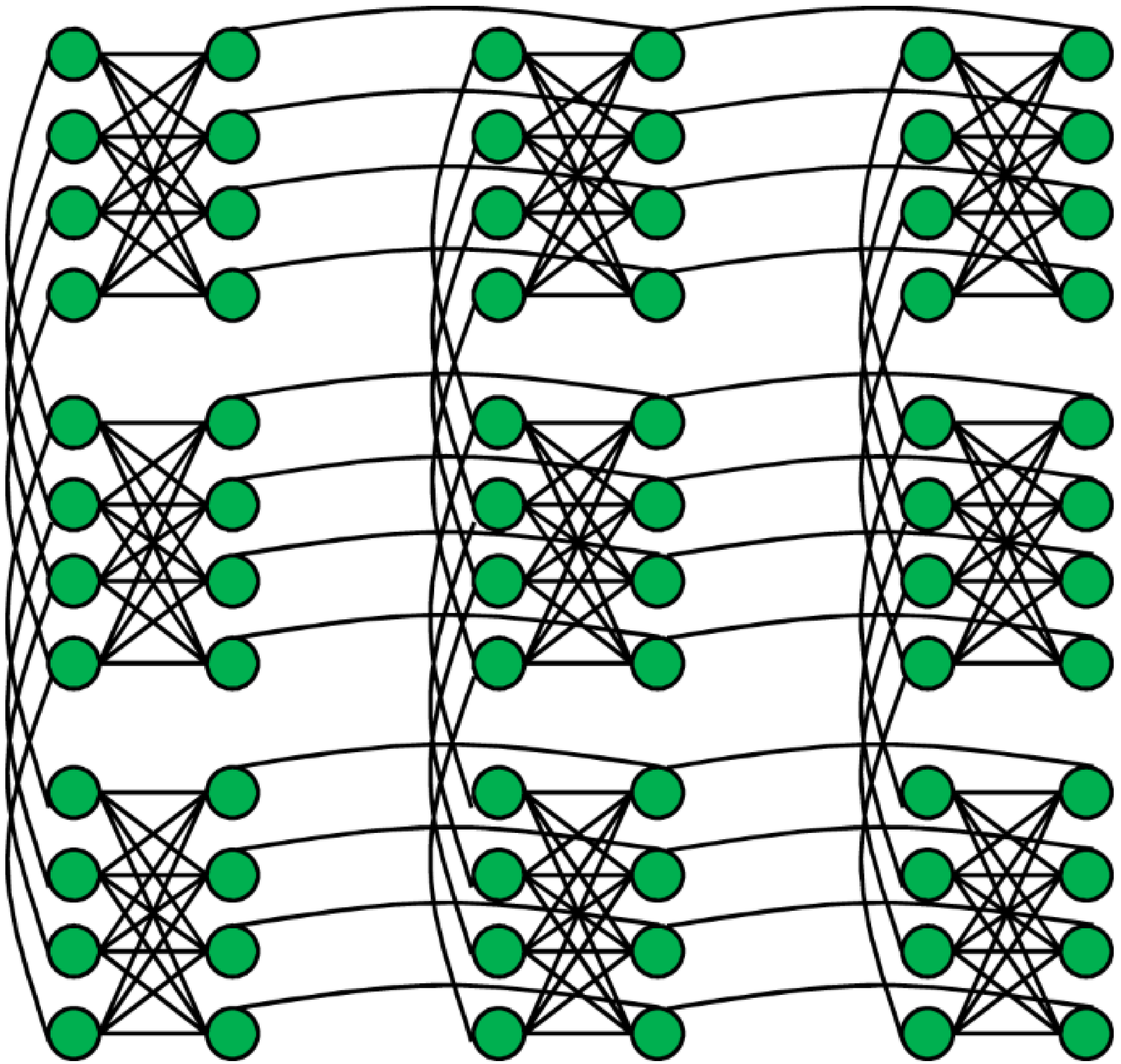

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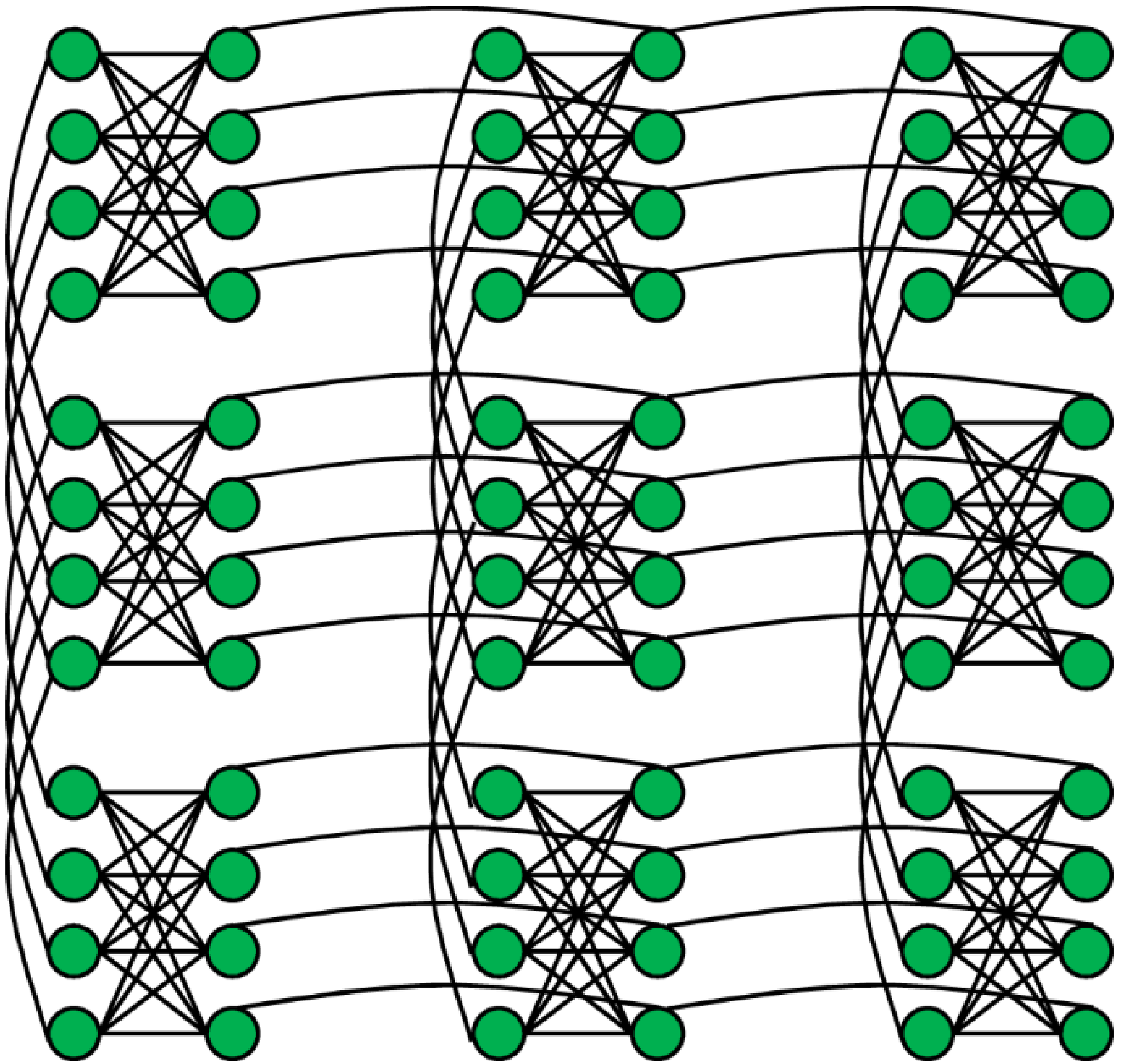

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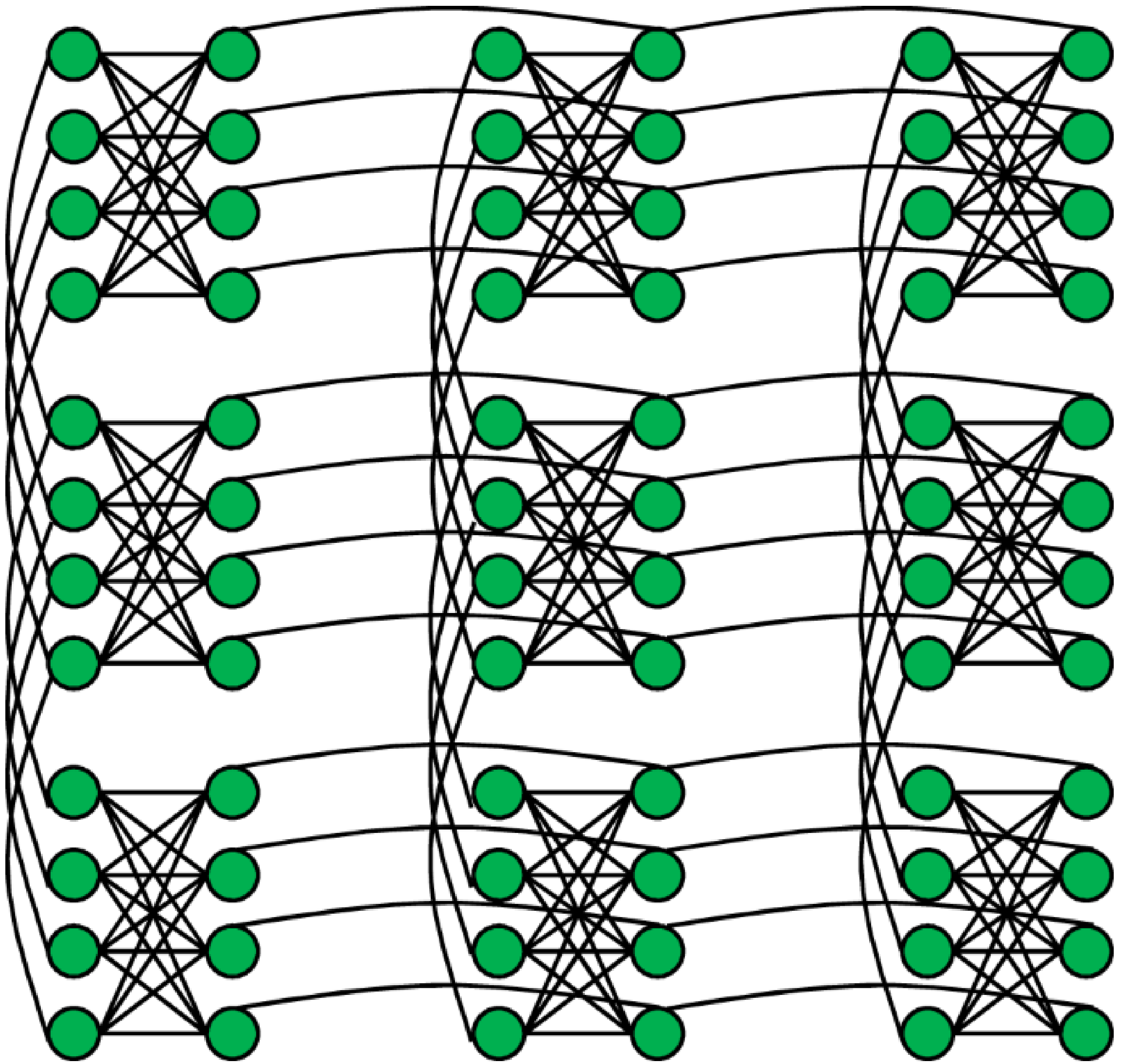

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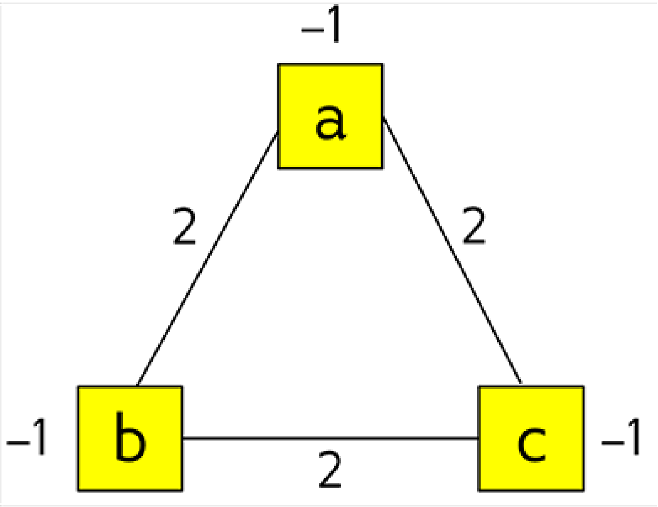
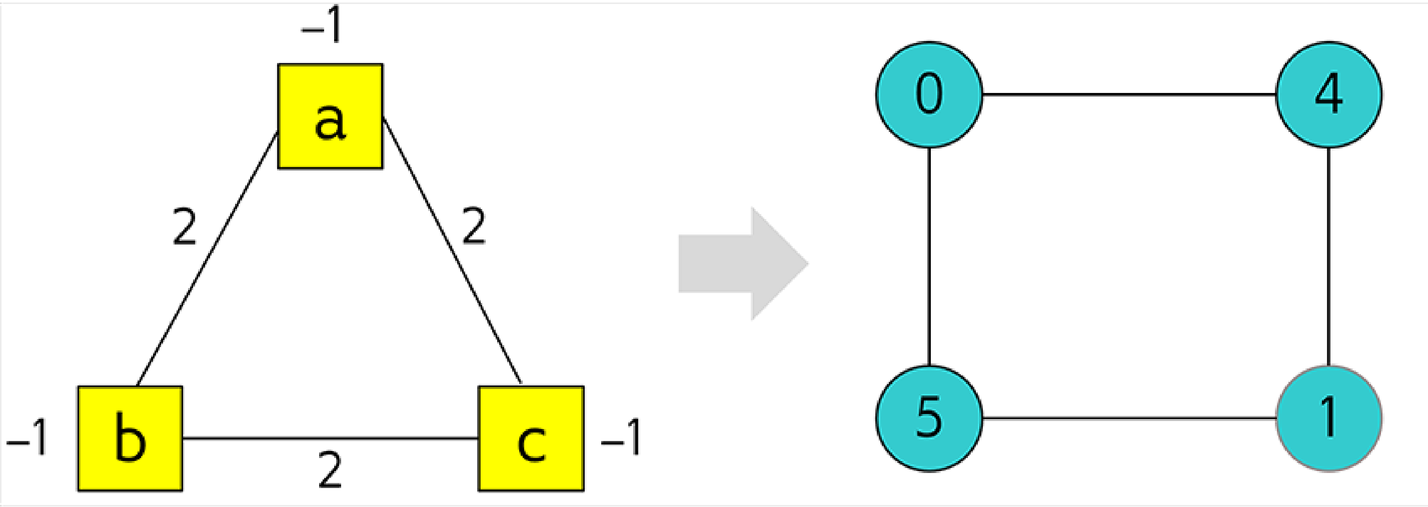


# Embedding a problem on a graph

- Suppose we have a QUBO problem that can be translated with the following graph
- Suppose we also have a quantum annealer with a graph of this shape



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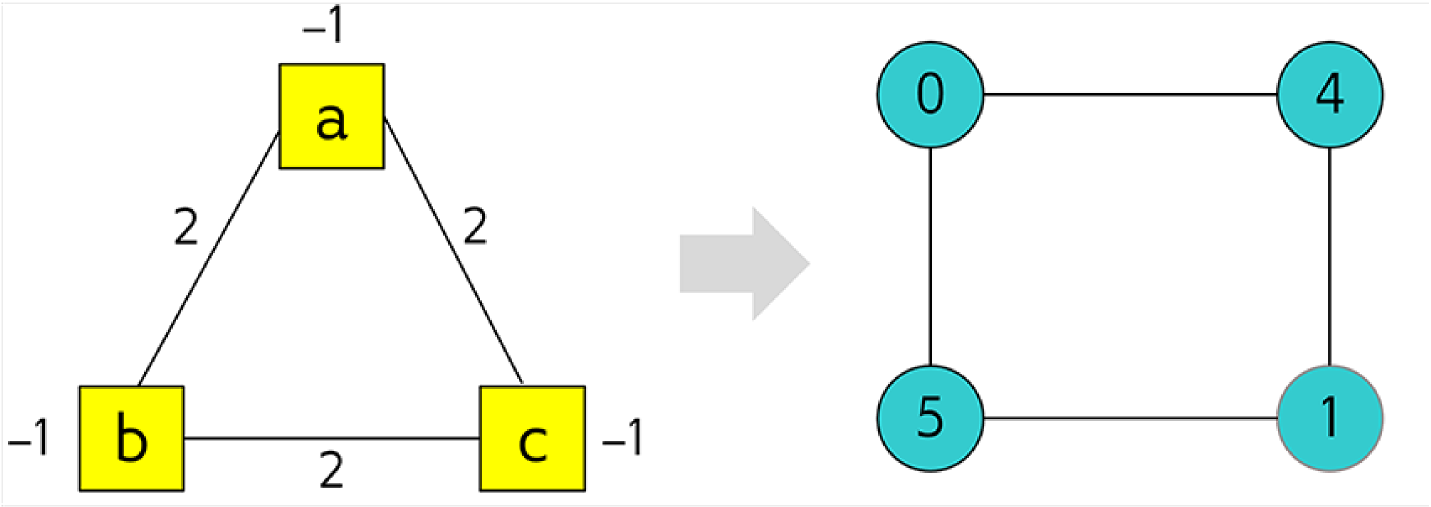

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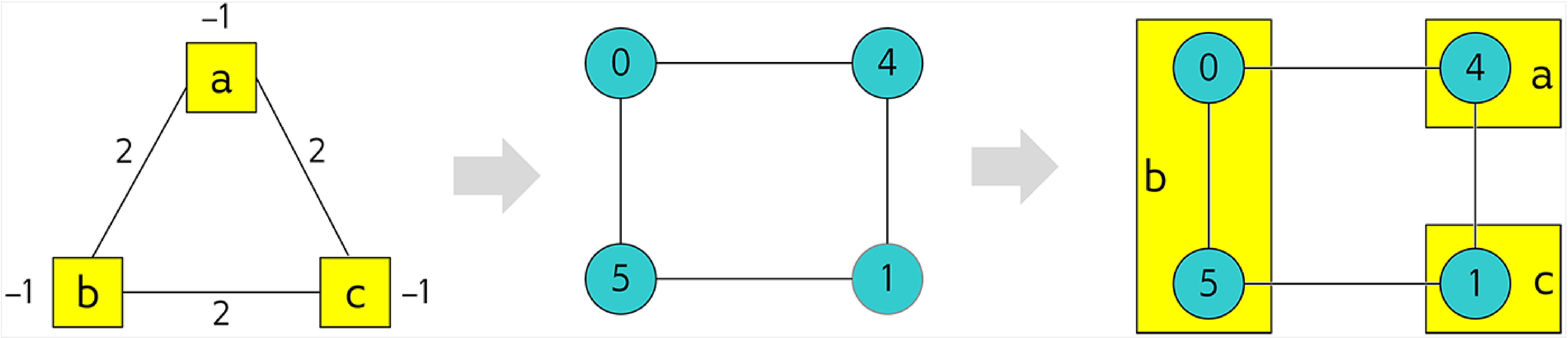

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- By looking at them, it seems impossible to map our problem to the target graph
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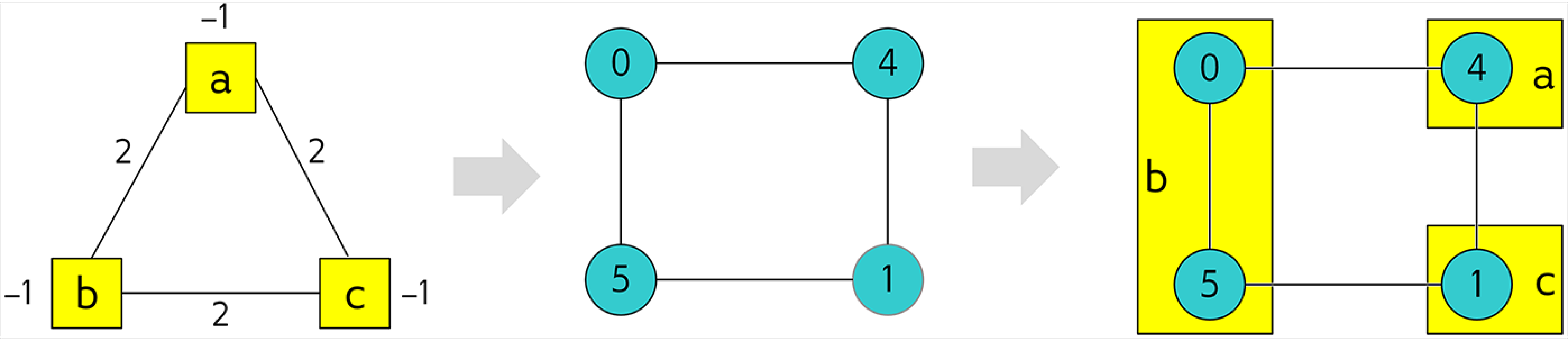
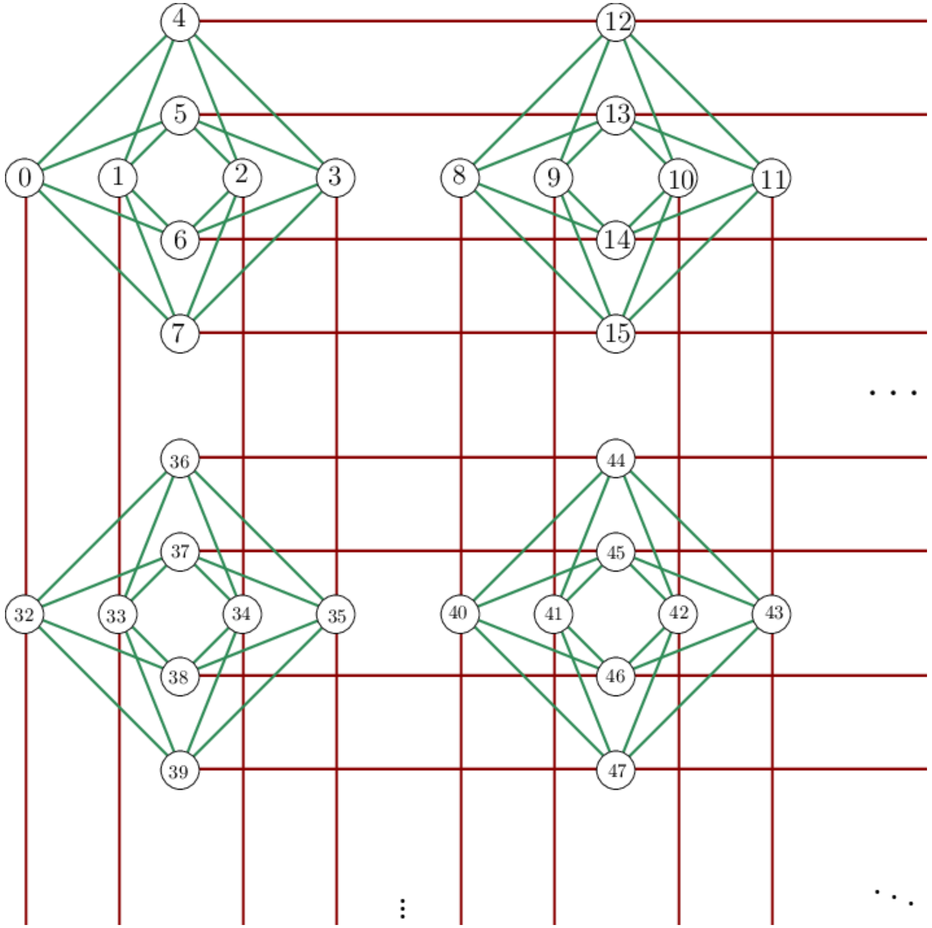
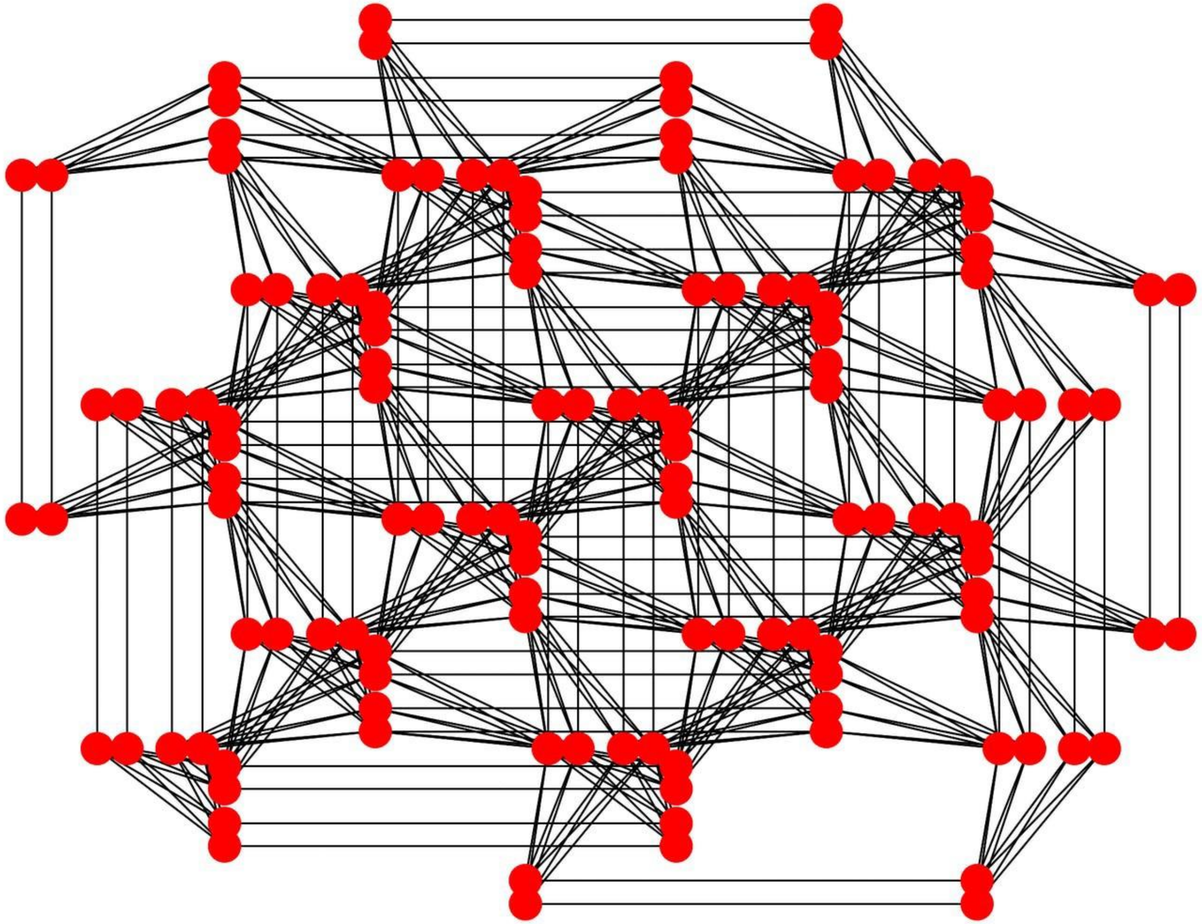
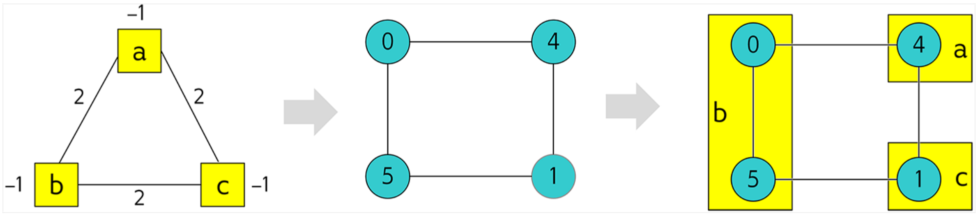
- In a certain sense, we can say that the qubits engaged in embedding are placed in entanglement relationship: they are forced to collapse in the same classical state



# Embedding on Chimera and Pegasus






CINECA QUANTUM COMPUTING LAB

# Exercise 2: Antenna Placement

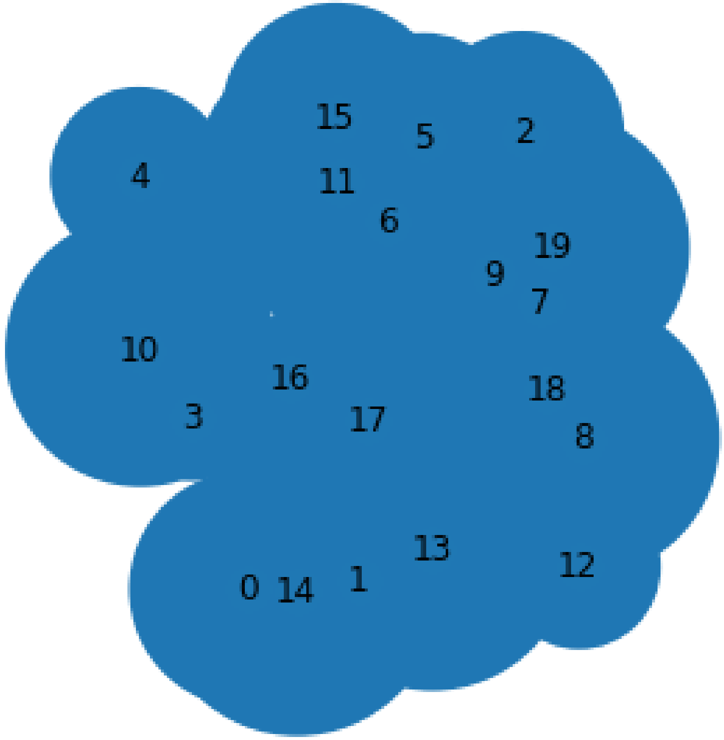
- Implement constraint into the Antenna Placement Problem

$$A_i = r_i^2 \cdot \pi \qquad B_{ij} = \rho_{ij}^2 \cdot \pi$$

$$QUBO = -\sum_{i=0}^{N} A_i q_i + \alpha \sum_{i<j} B_{ij} q_i q_j$$

$$\min\left(\left(\sum_{i=0}^{N} \beta(1 - 2F) q_i + \sum_{i<j} 2\beta q_i q_j\right)\right) - \sum_{i=1}^{N} \gamma L_i q_i \qquad \sum_{i=1}^{N+F} q_i = F$$

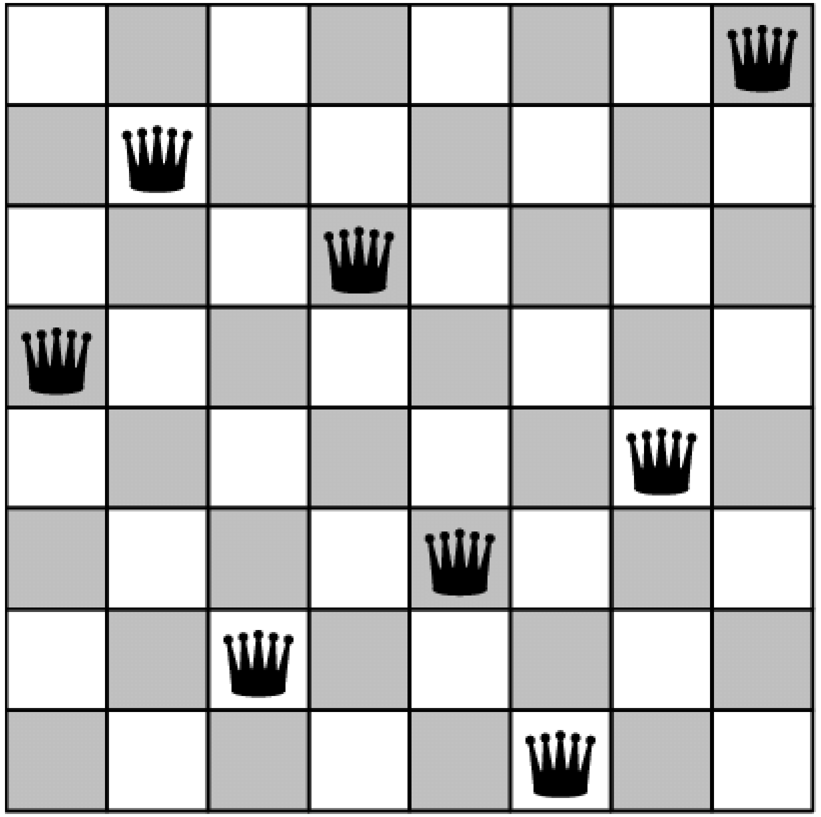
$$\sum_{i=1}^{2N} C_i A_i q_i = A_m$$




- Try the embedding on Pegasus and Chimera

# Exercise 3: N-Queens Puzzle

- Let us now turn to another problem: **the N-queens puzzle**



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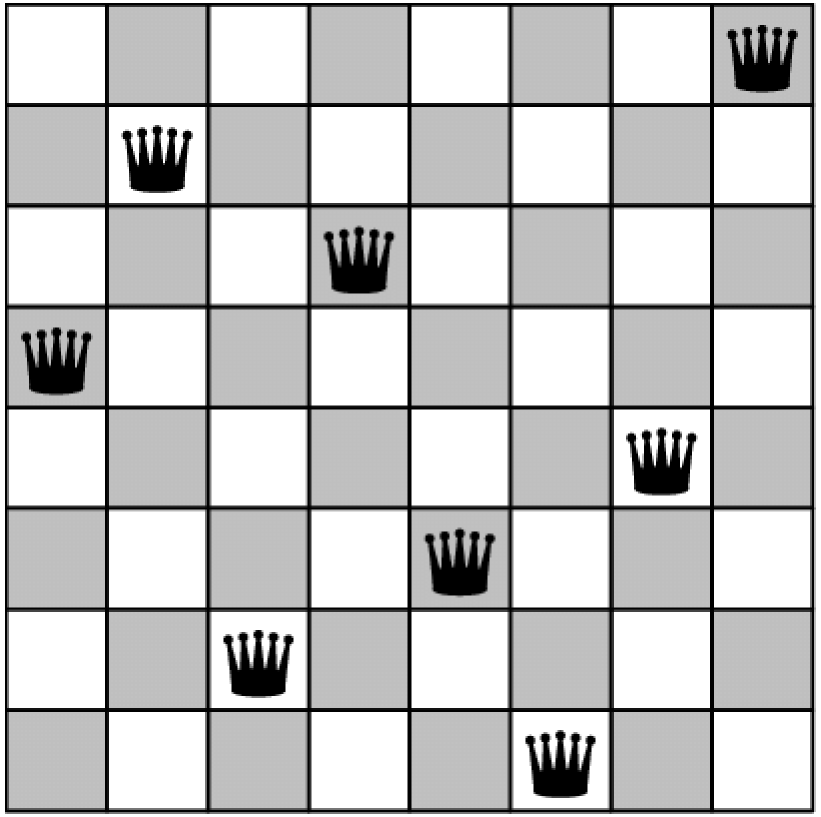

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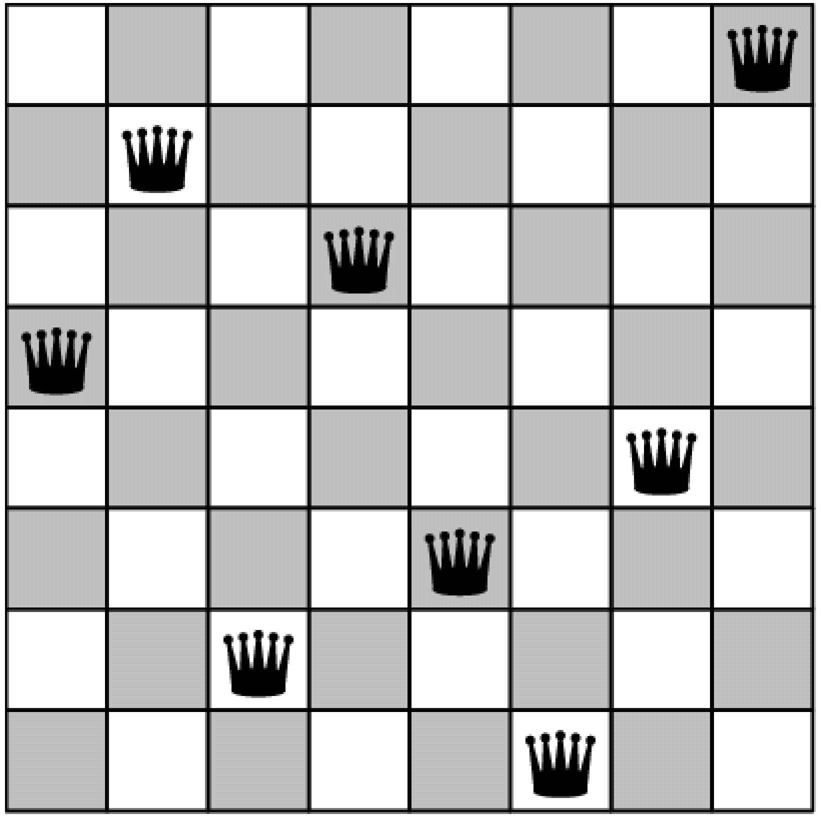

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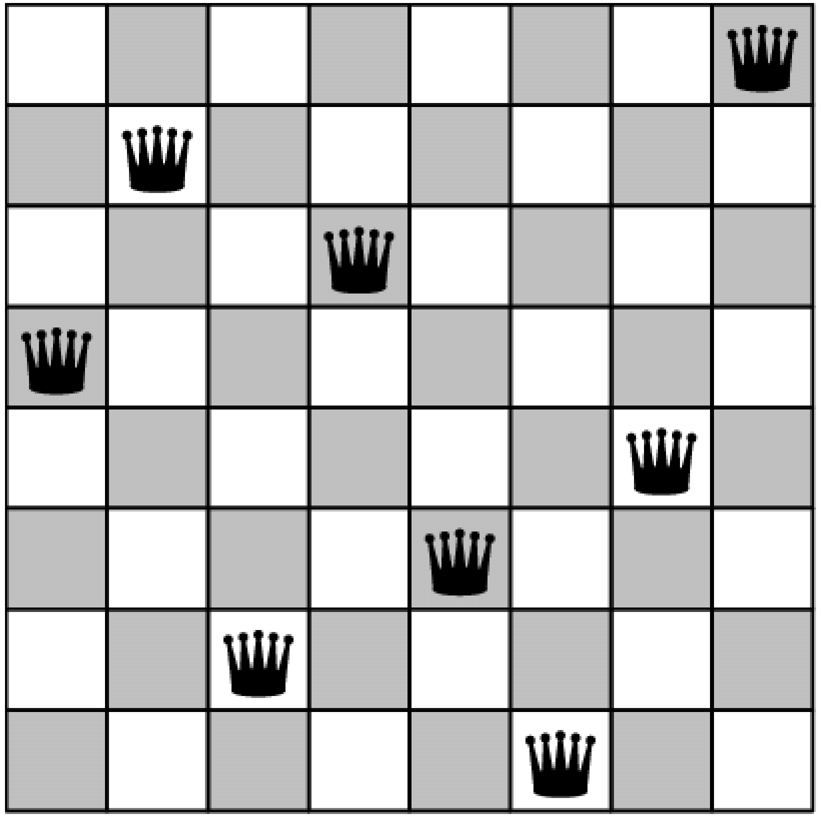
- The game is generalized as follows: let's consider a chessboard **of dimension NxN**. Find a way **to arrange N queens** on the chessboard so that none of them are in check by any of the other queens



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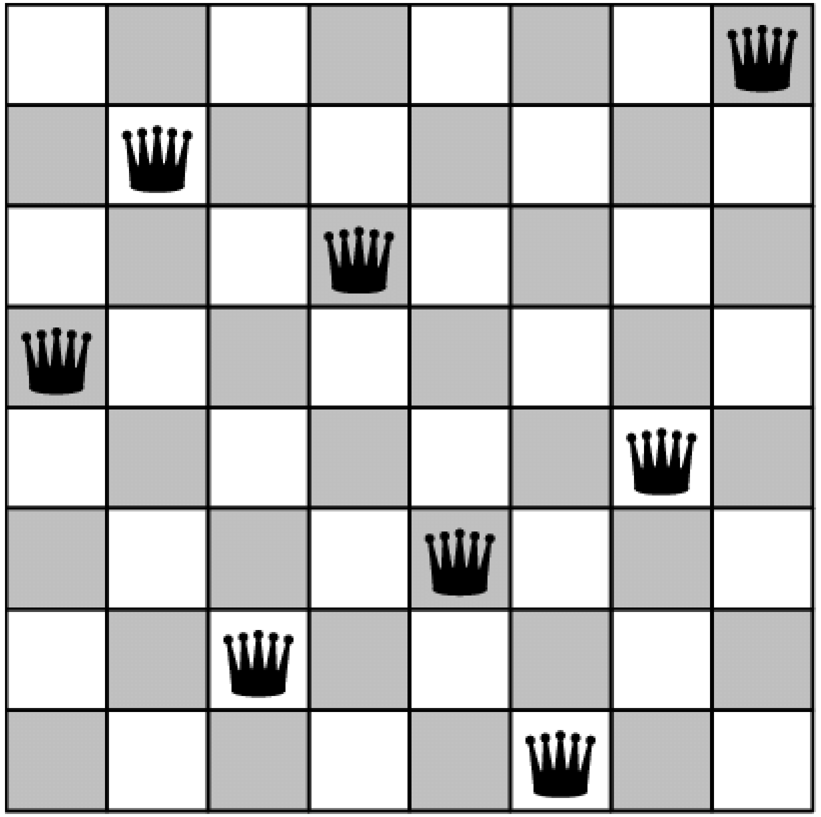
- Let's think about how to turn the problem into a QUBO problem



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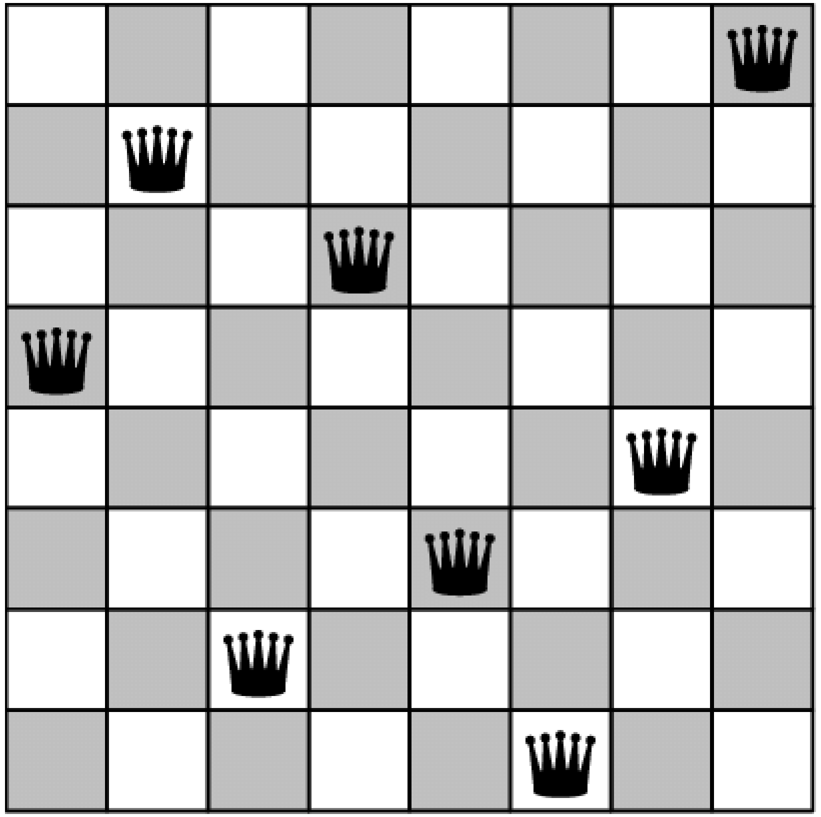
- Let's think about how to turn the problem into a QUBO problem
- We therefore consider a **vector of binary variables**. We will take **one for each square** of the board we are considering.



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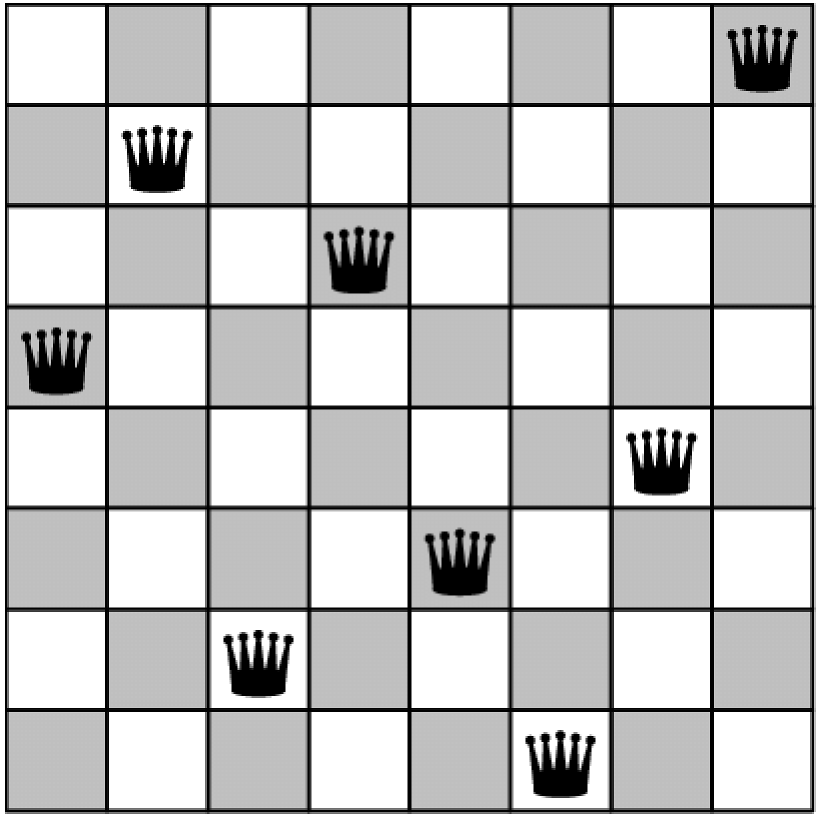

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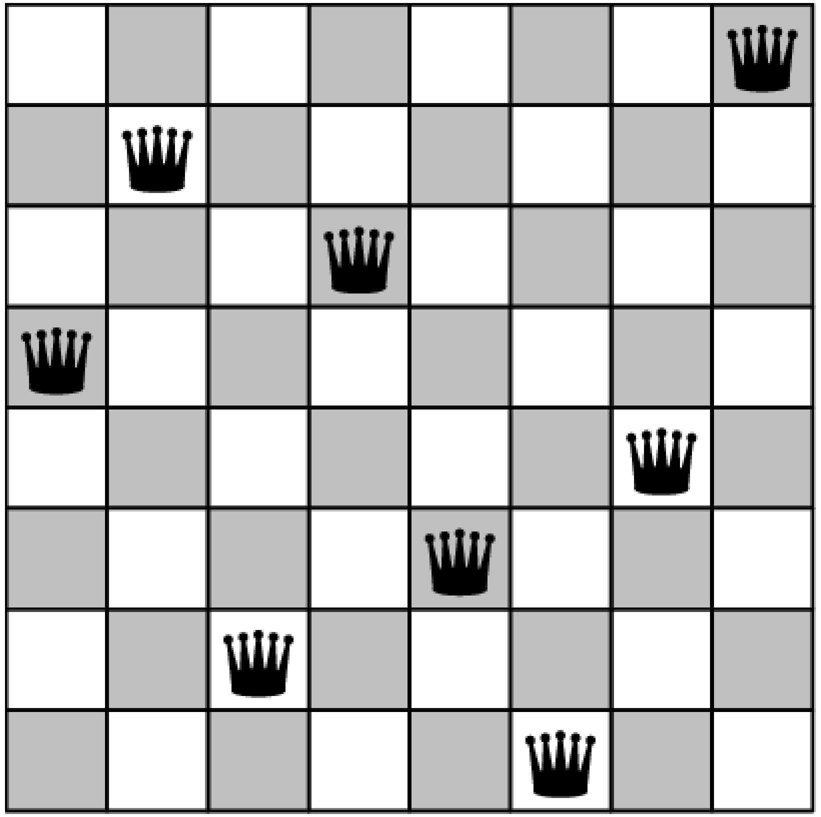

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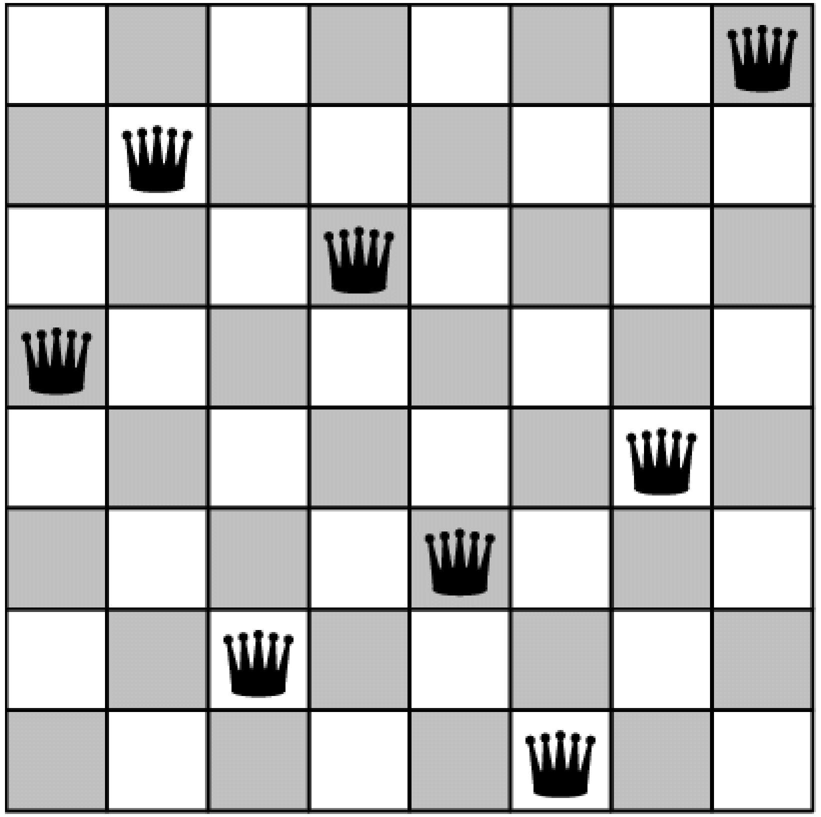

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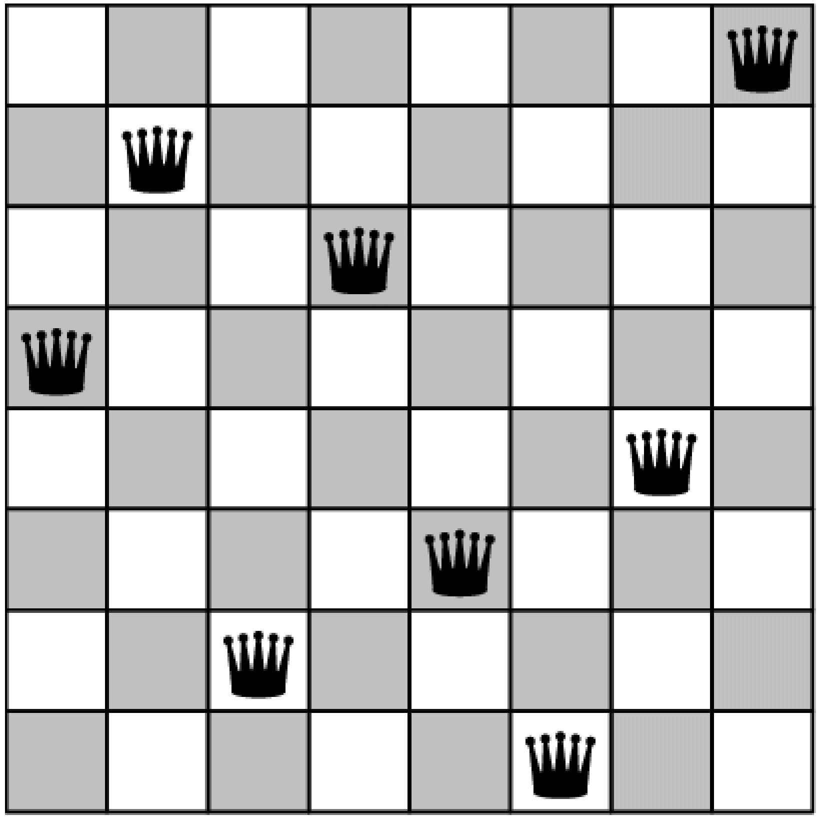

$$\sum_{i=1}^{N} q_i = N$$

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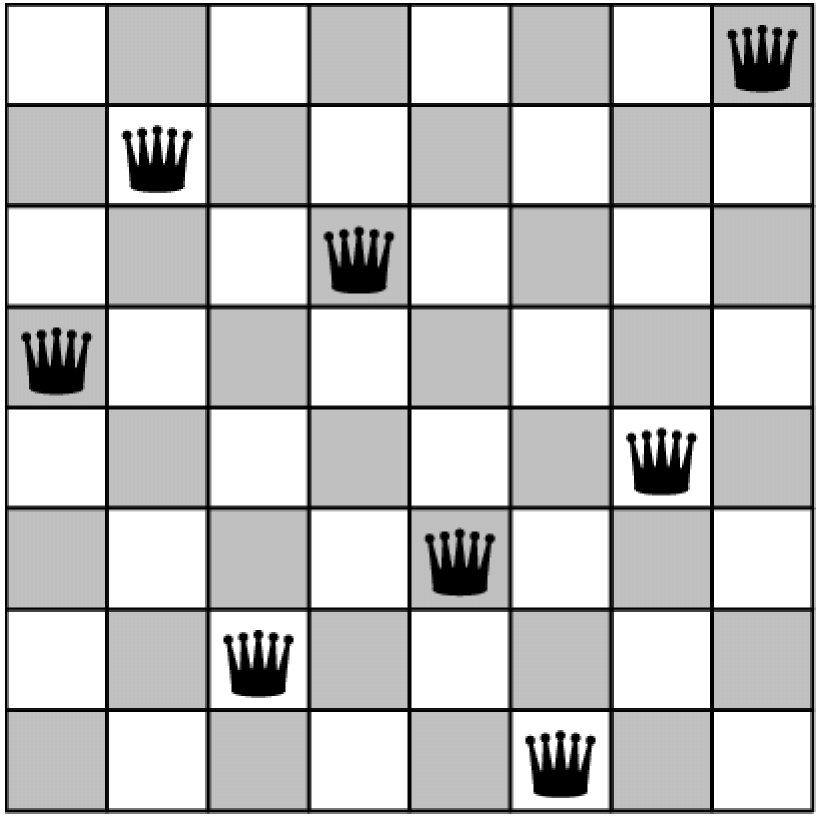
$$\sum_{i=1}^{N} q_i = N \Rightarrow \min \left( \sum_{i=1}^{N} q_i - N \right)^2$$



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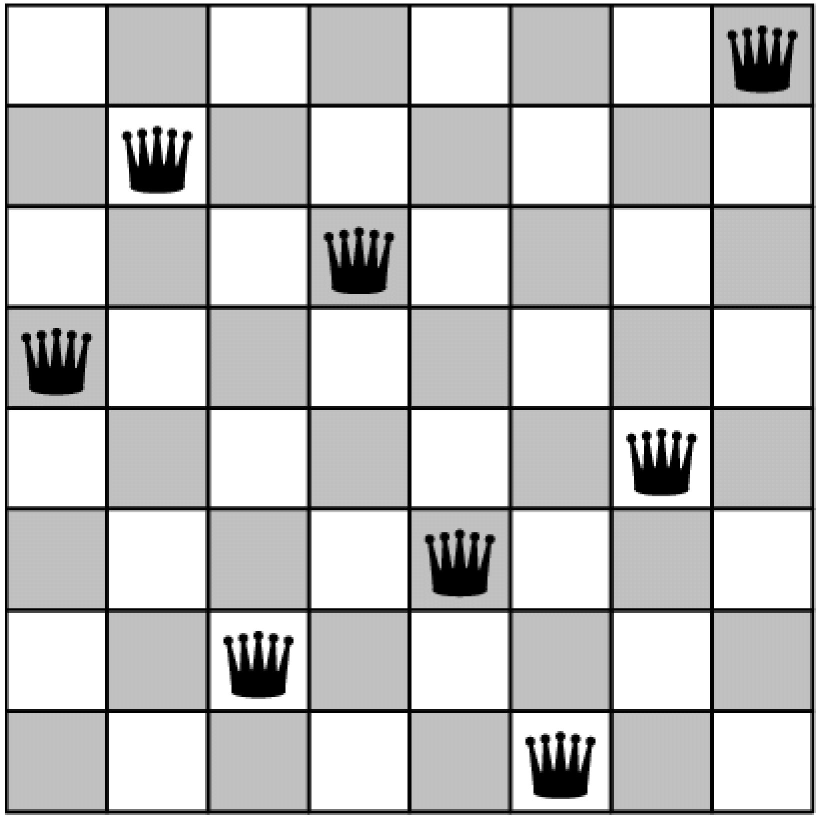
$$\left(\sum_{i=1}^{N} q_i - N\right)^2$$



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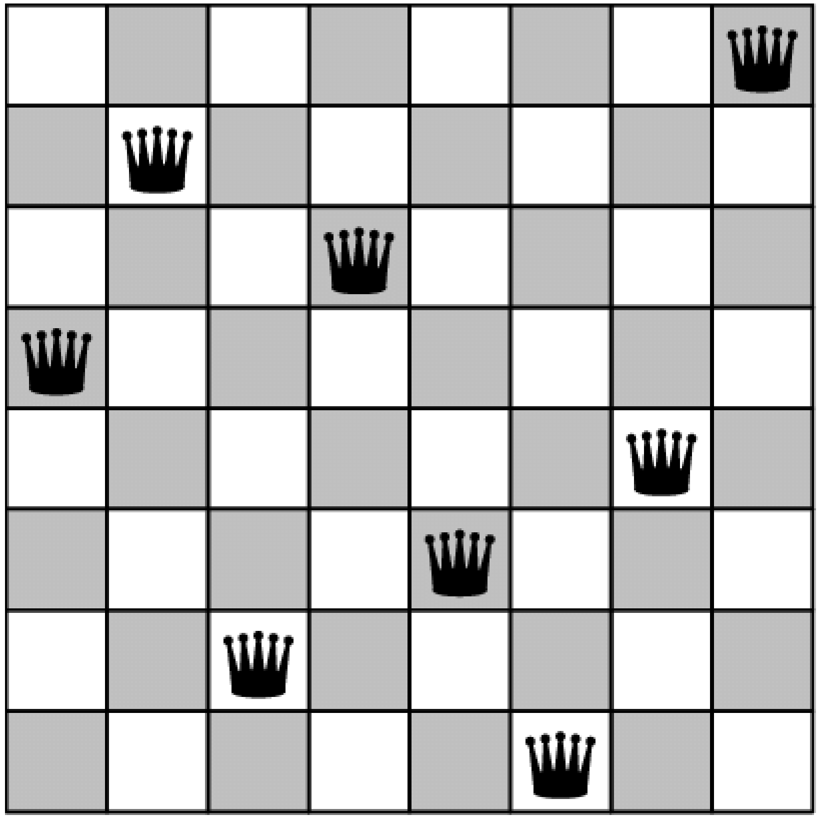

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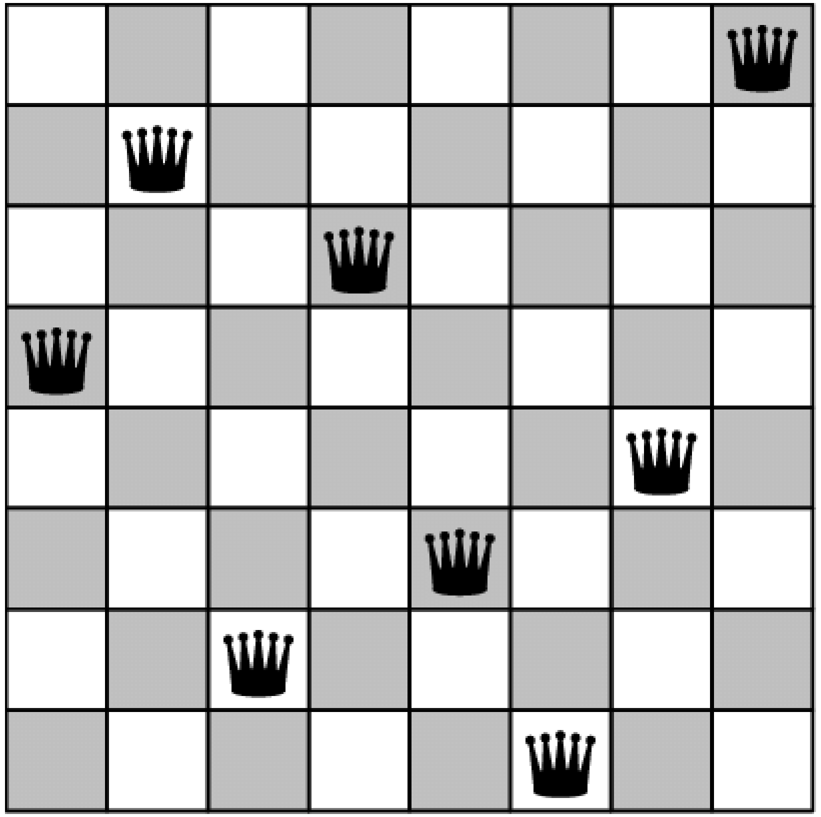

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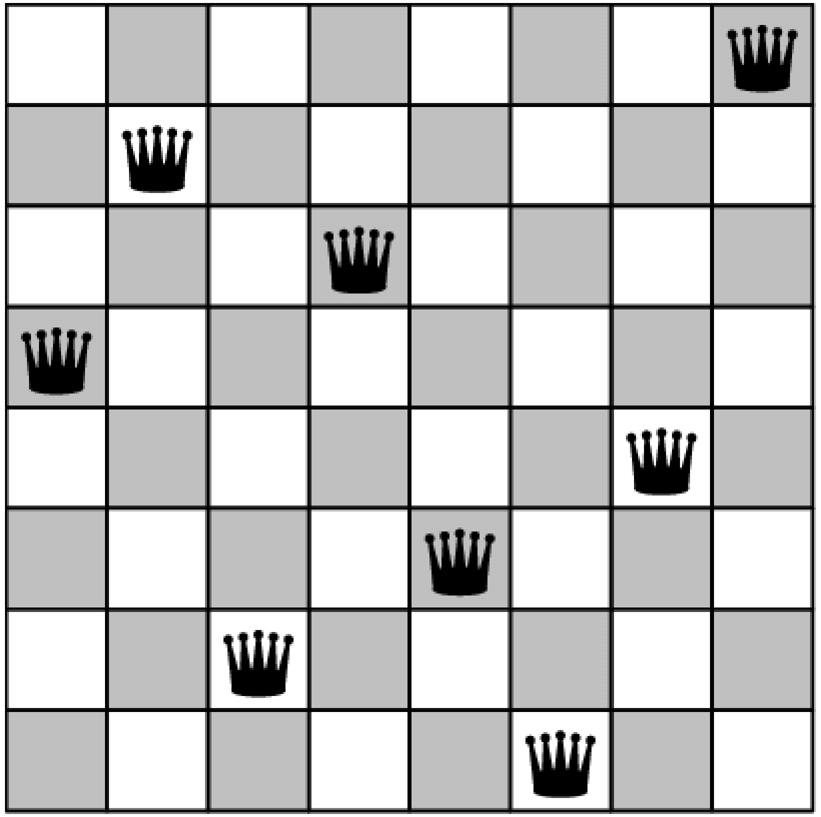

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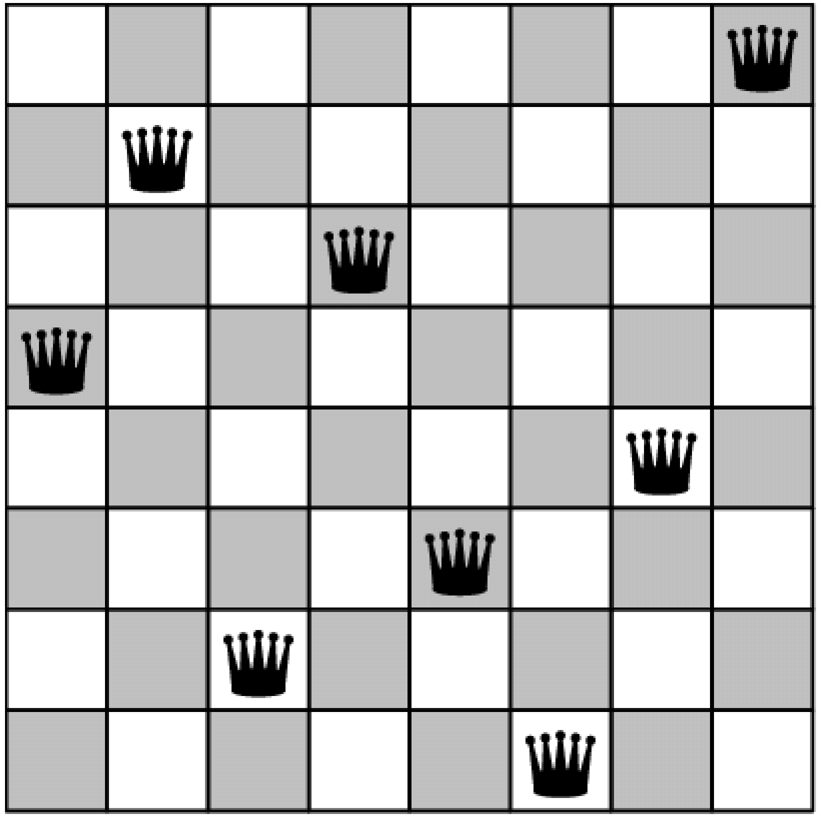

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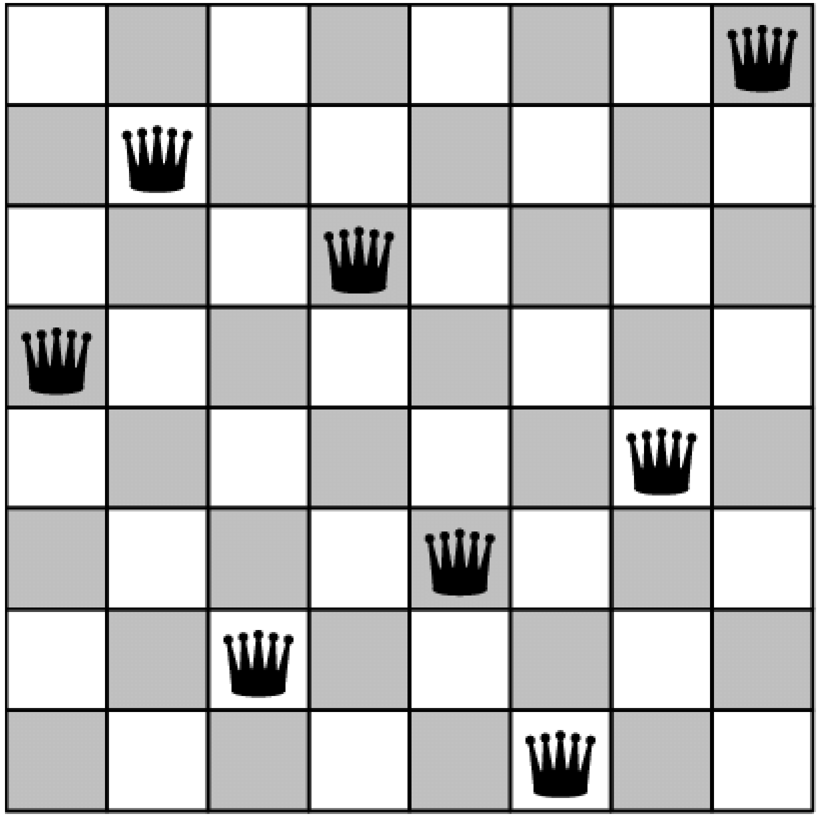

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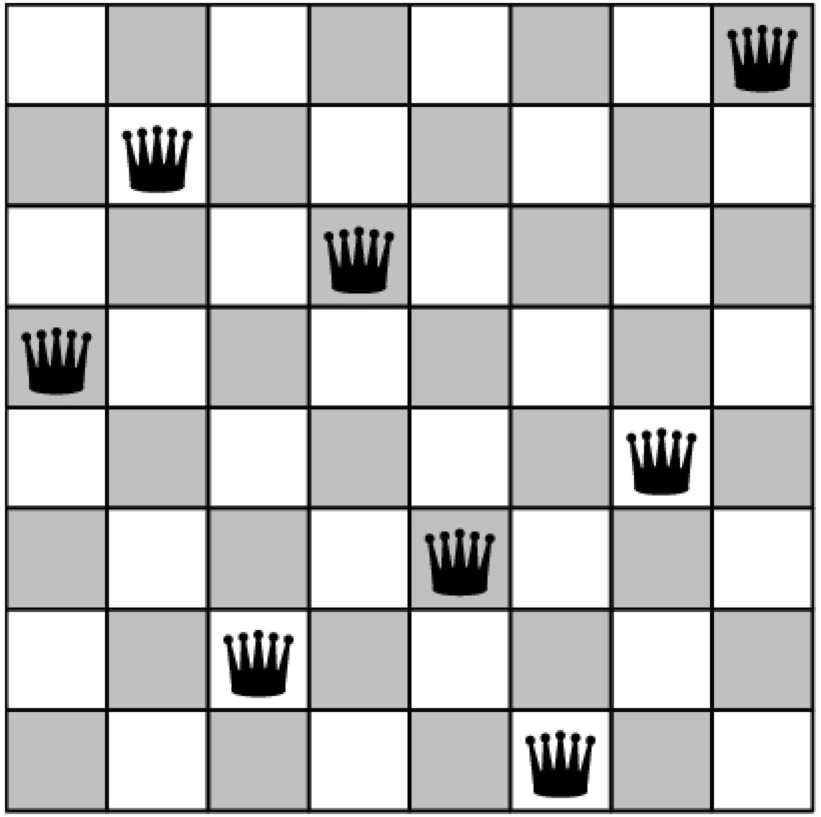

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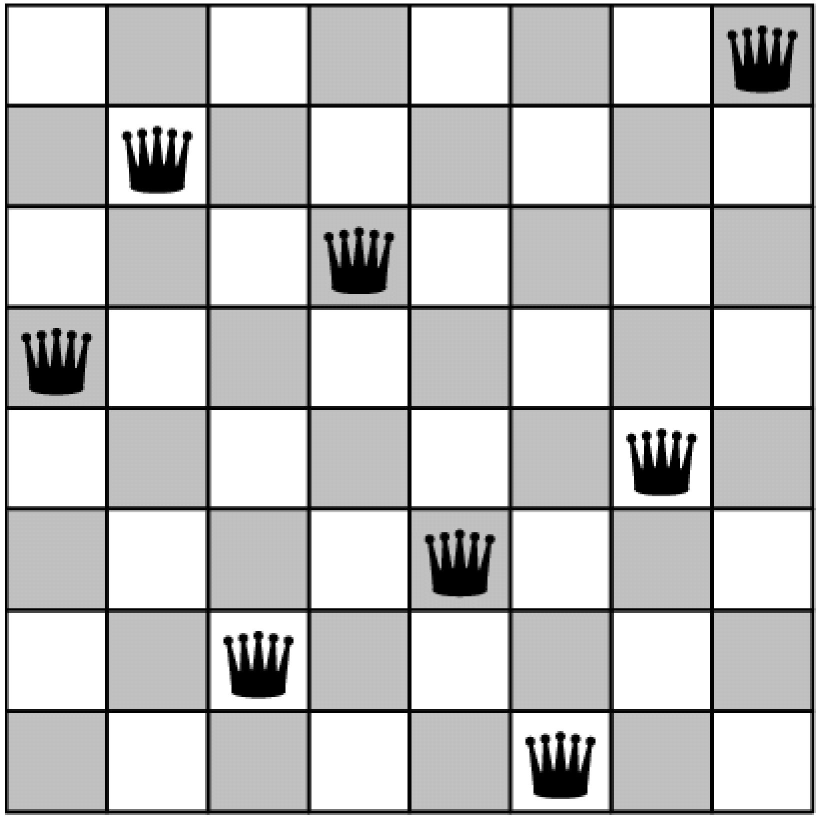

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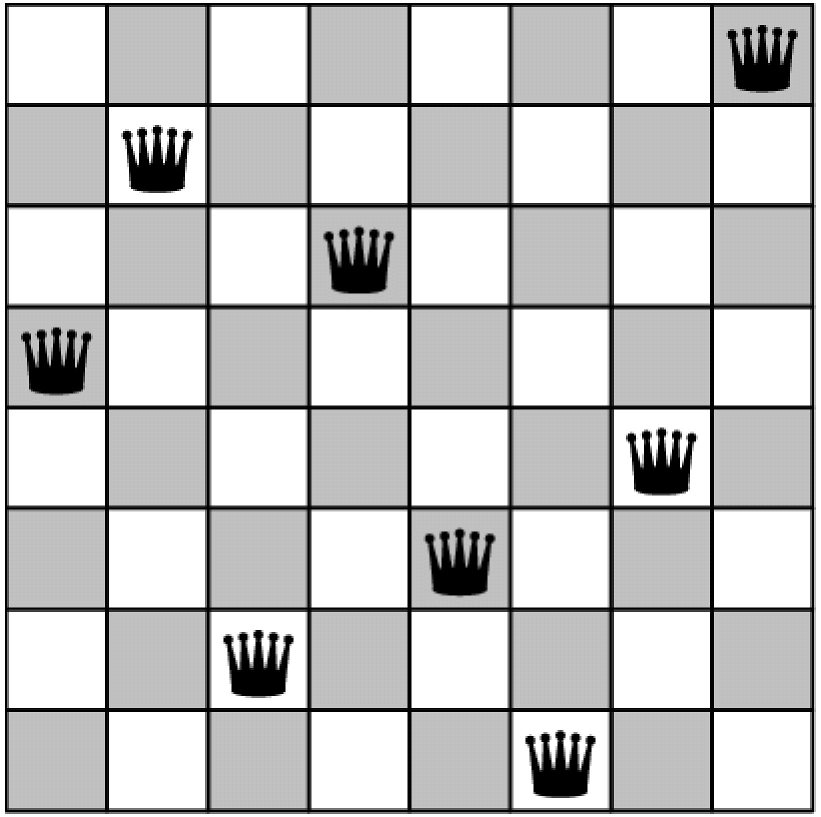

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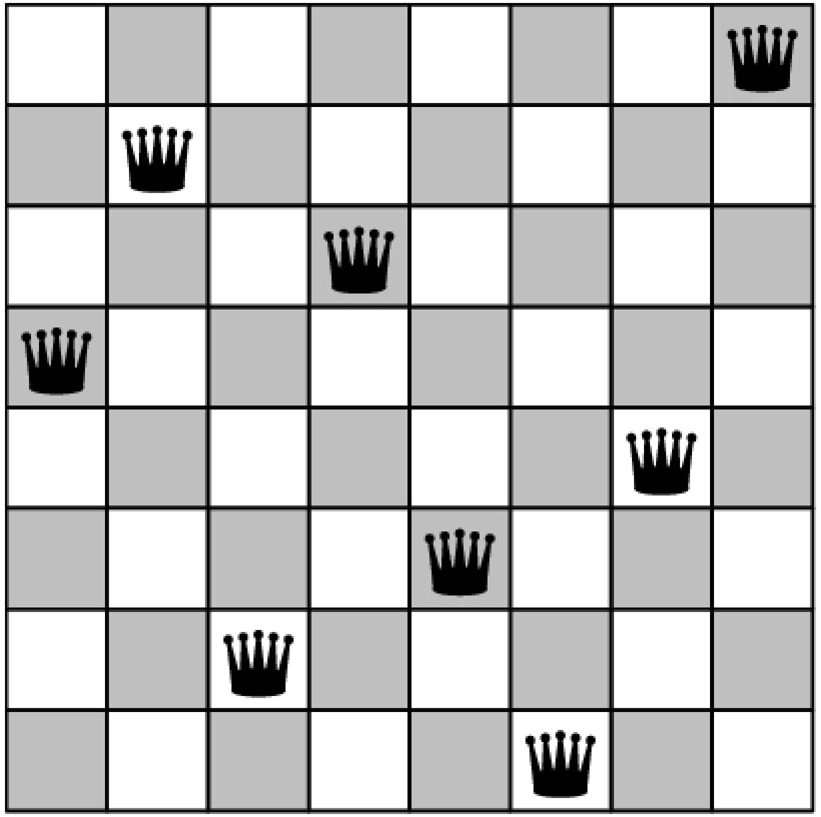

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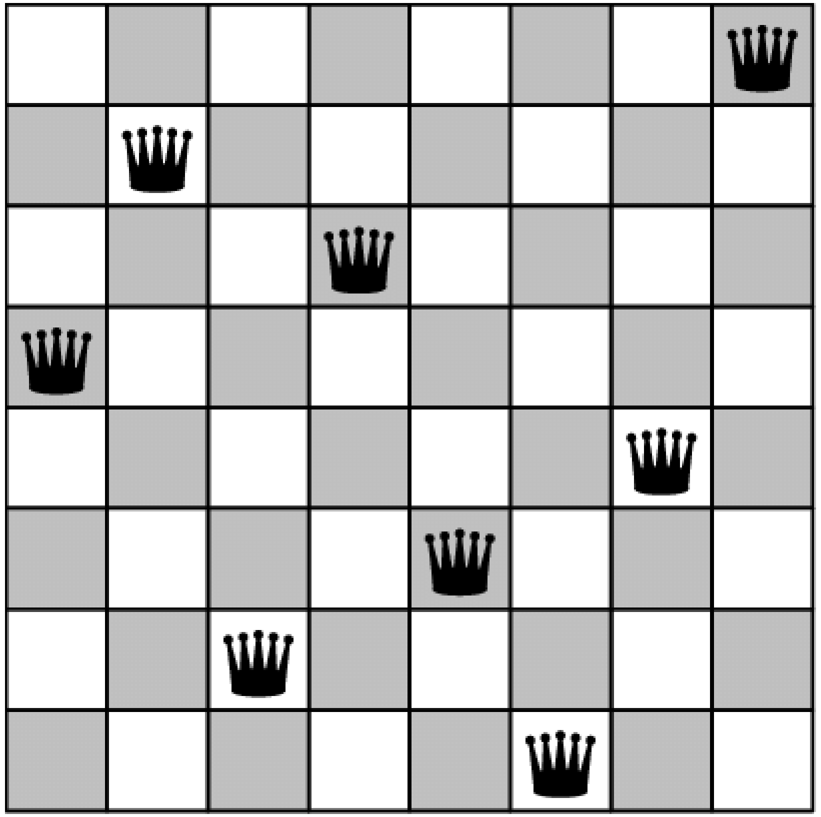

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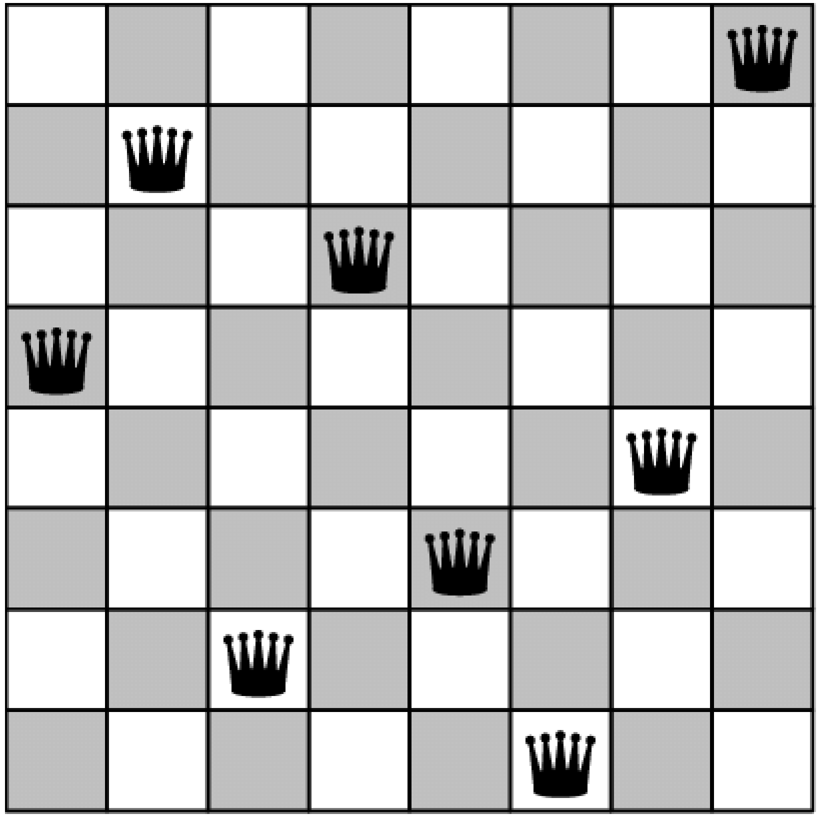

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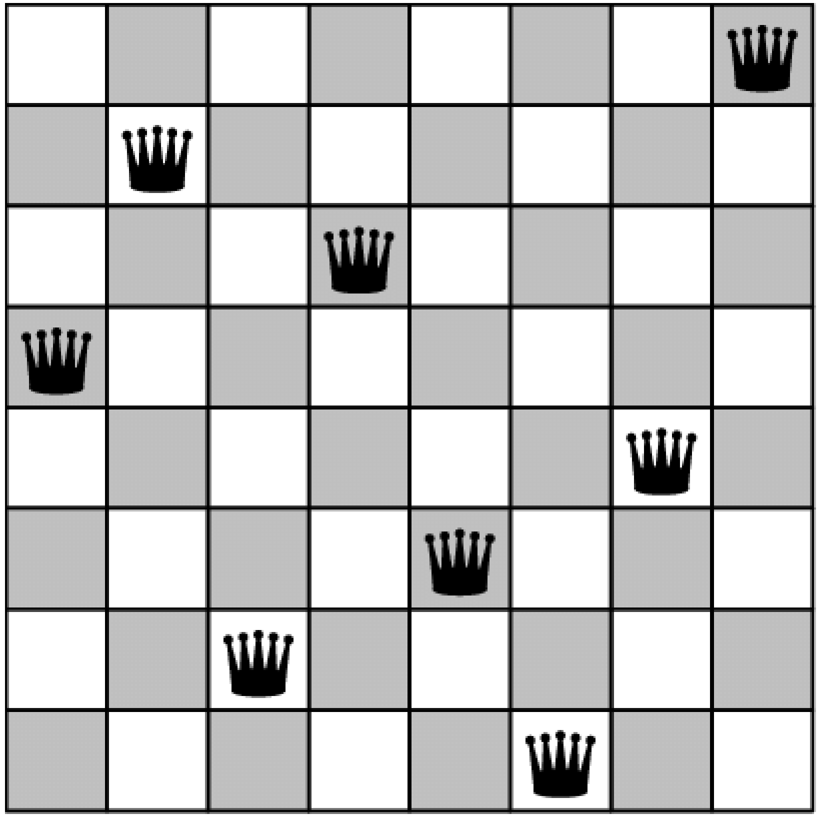

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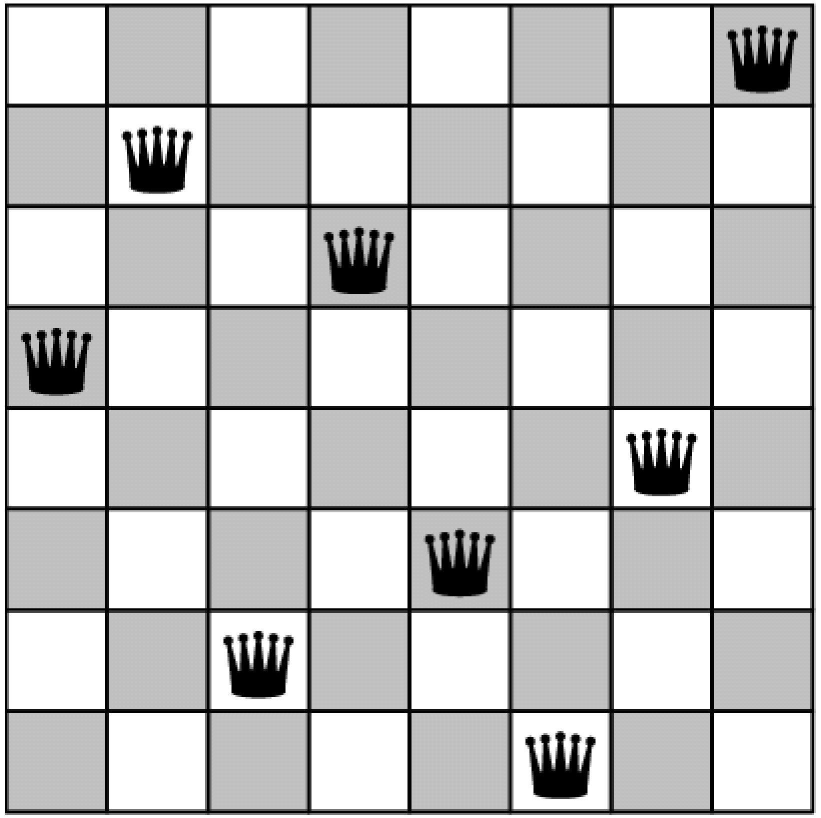

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- Let us then consider the problem from another point of view.
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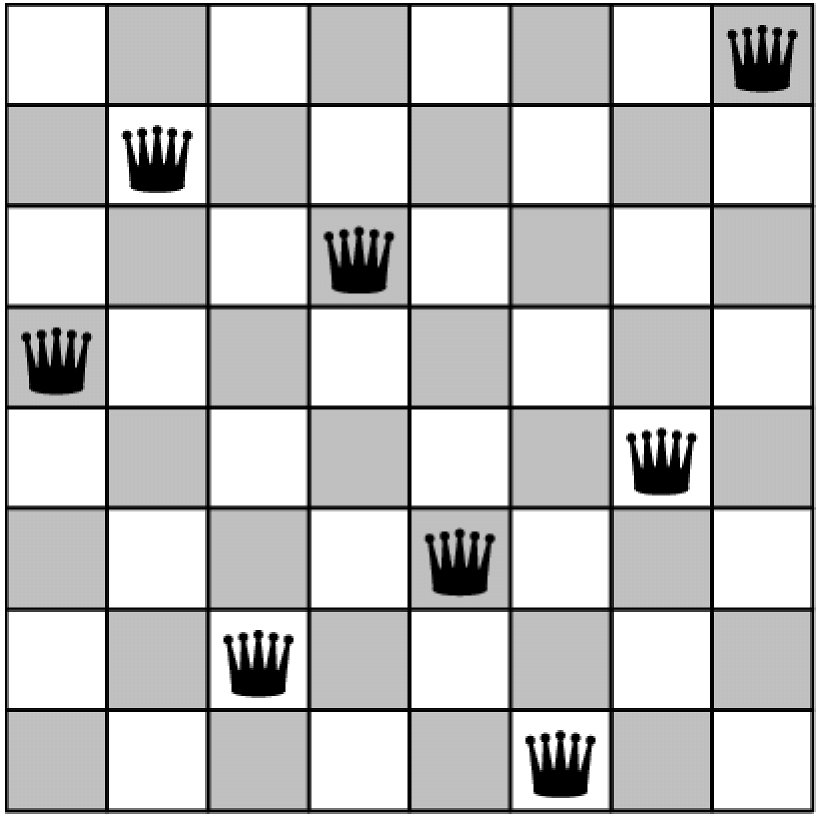
- Let us consider the matrix of the quadratic contributions of the QUBO problem



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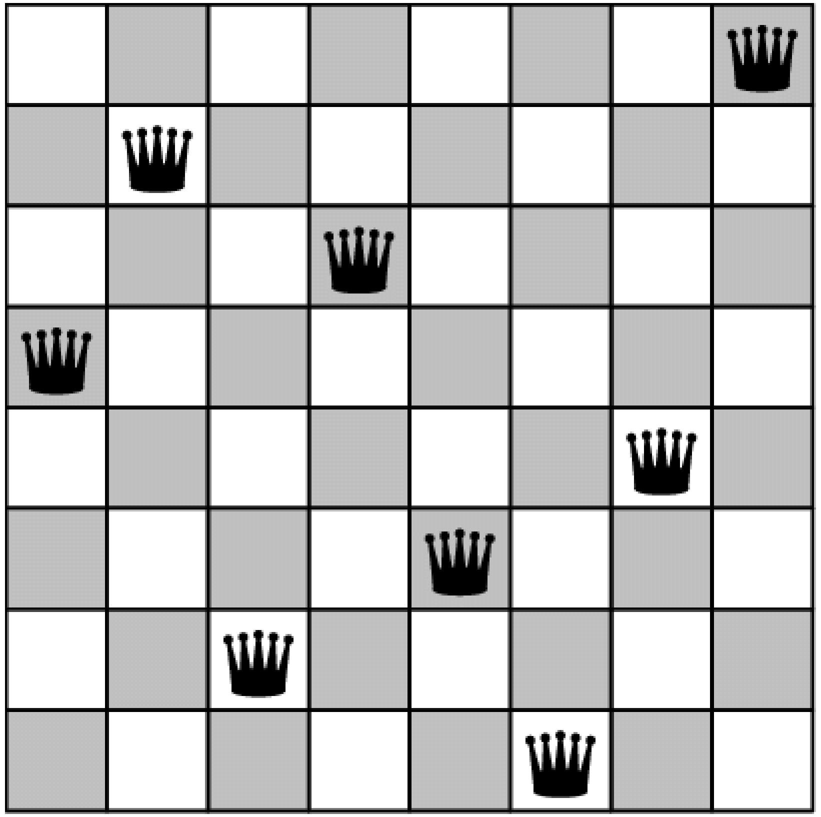

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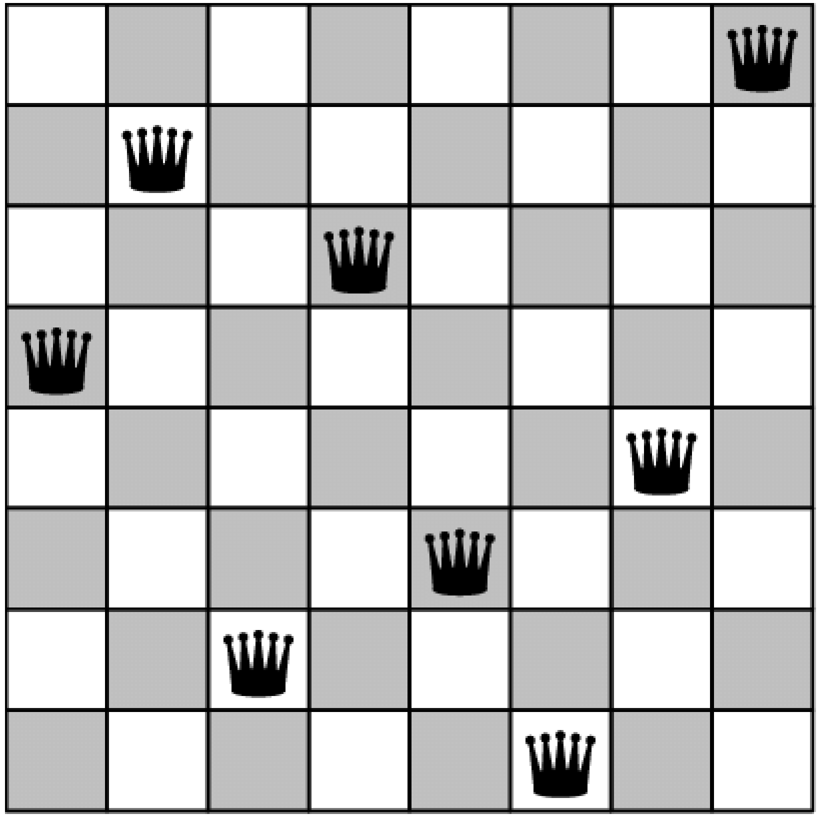

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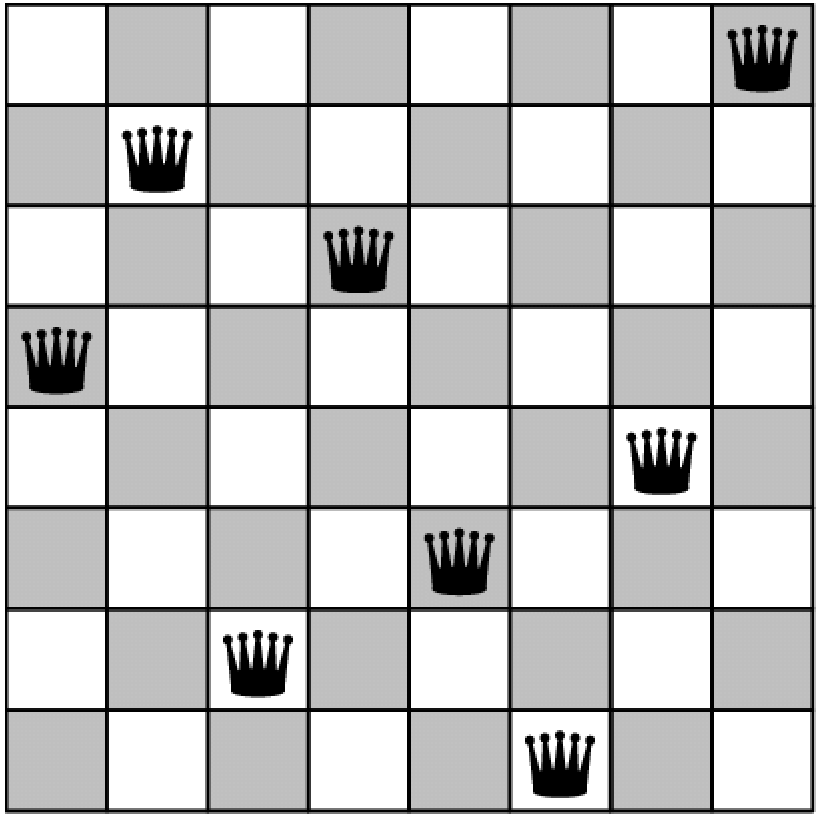
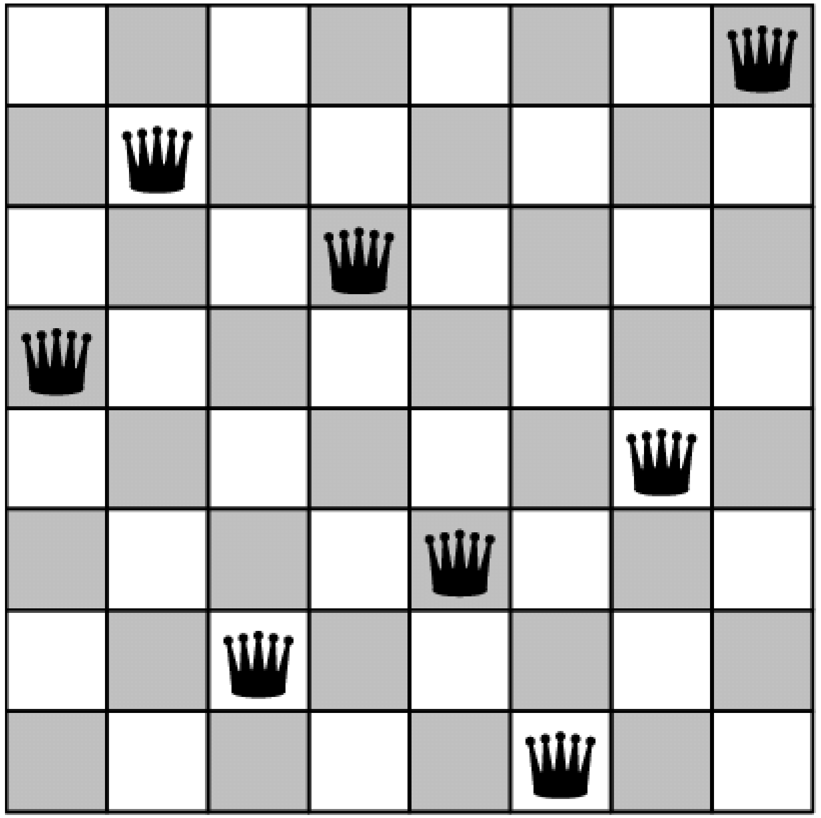
- To implement our constraints, we do the following: we analyze the matrix of the quadratic contributions and, for each pair that is "forbidden", we increase the value of its weight
- The weight, by definition, is activated only if both qubits, or squares, are in state 1, i.e. both host a queen



# Exercise 3: N-Queens Puzzle

- One way to do this, is to define a function in this way

```
def ROW(row,dim):
    C=np.zeros((dim,dim), dtype=int)
    C[row,:] = 1
    return C.flatten()
```



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def ROW(row,dim):
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    C[row,:] = 1
    return C.flatten()
```
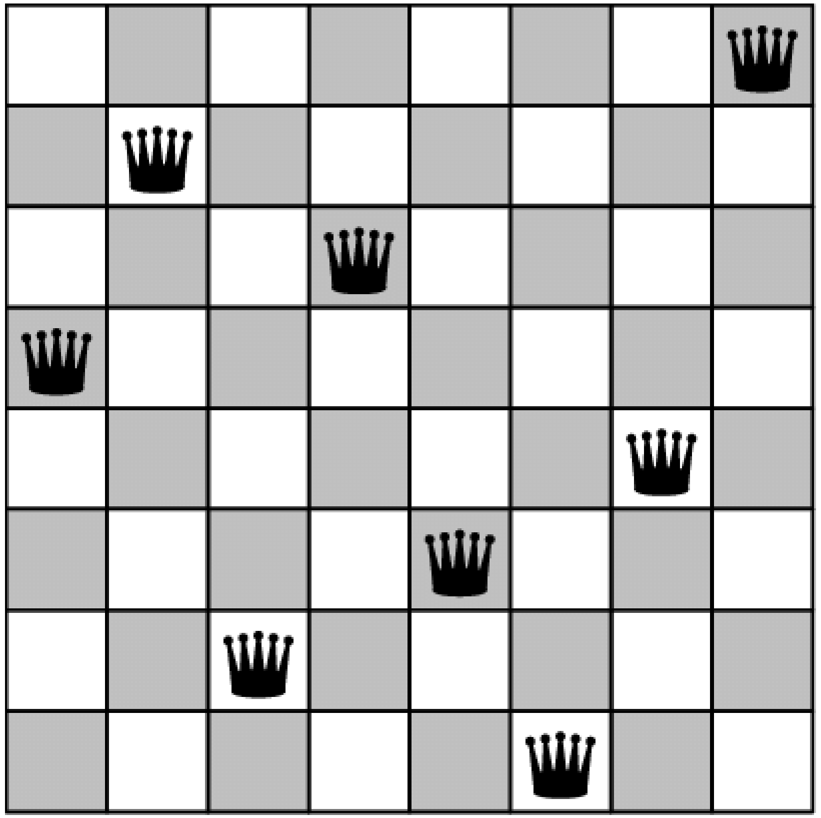

- This function calculates a linearized vector containing all the possible pairs of squares on the board

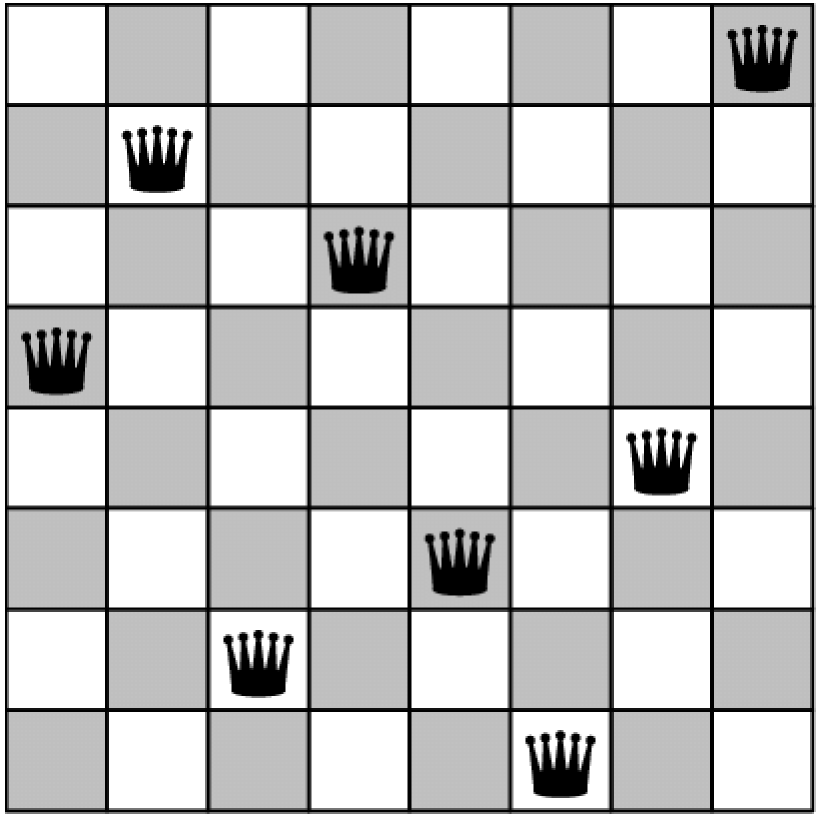


# Exercise 3: N-Queens Puzzle

- One way to do this, is to define a function in this way

```
def ROW(row,dim):
    C=np.zeros((dim,dim), dtype=int)
    C[row,:] = 1
    return C.flatten()
```

- This function calculates a linearized vector containing all the possible pairs of squares on the board
- Once a specific row has been chosen, the vector will have the value 1 if the pair of squares belongs to the same row, 0 otherwise



# Exercise 3: N-Queens Puzzle

- One way to do this, is to define a function in this way

```python
def ROW(row,dim):
    C=np.zeros((dim,dim), dtype=int)
    C[row,:] = 1
    return C.flatten()
```

- This function calculates a linearized vector containing all the possible pairs of squares on the board
- Once a specific row has been chosen, the vector will have the value 1 if the pair of squares belongs to the same row, 0 otherwise
- With this definition, I can start building the penalty matrix like this

```python
N=4

w=1

B=np.zeros((N*N,N*N), dtype=float)
for row in range(N):
    R=ROW(row,N)
    for i in range(N*N):
        for j in range(i+1,N*N):
            B[i][j]=B[i][j]+R[i]*R[j]*w
print(B)
```

```
[[0. 1. 1. 1. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 1. 1. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 1. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 1. 1. 1. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 1. 1. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 1. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]]
```

# Exercise 3: N-Queens Puzzle

- One way to do this, is to define a function in this way

```
def ROW(row,dim):
    C=np.zeros((dim,dim), dtype=int)
    C[row,:] = 1
    return C.flatten()
```

- This function calculates a linearized vector containing all the possible pairs of squares on the board
- Once a specific row has been chosen, the vector will have the value 1 if the pair of squares belongs to the same row, 0 otherwise
- With this definition, I can start building the penalty matrix like this
- Basically I'm saying: if two squares are part of the same row, it increases their weight by a factor w

```
N=4

w=1

B=np.zeros((N*N,N*N), dtype=float)
for row in range(N):
    R=ROW(row,N)
    for i in range(N*N):
        for j in range(i+1,N*N):
            B[i][j]=B[i][j]+R[i]*R[j]*w
print(B)
```

```
[[0. 1. 1. 1. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 1. 1. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 1. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 1. 1. 1. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 1. 1. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 1. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]]
```

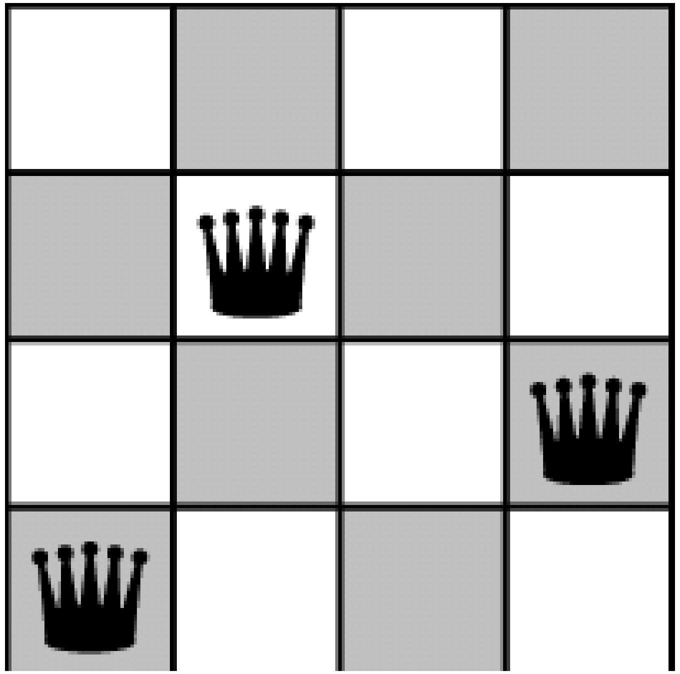
# Exercise 3: N-Queens Puzzle

- One way to do this, is to define a function in this way

```
def ROW(row,dim):
    C=np.zeros((dim,dim), dtype=int)
    C[row,:] = 1
    return C.flatten()
```

```
R=ROW(0,4)
```

```
[[1 1 1 1]
 [0 0 0 0]
 [0 0 0 0]
 [0 0 0 0]]
```



```
N=4

w=1

B=np.zeros((N*N,N*N), dtype=float)
for row in range(N):
    R=ROW(row,N)
    for i in range(N*N):
        for j in range(i+1,N*N):
            B[i][j]=B[i][j]+R[i]*R[j]*w
print(B)
```

```
[[0. 1. 1. 1. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 1. 1. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 1. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 1. 1. 1. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 1. 1. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 1. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]]
```

# Exercise 3: N-Queens Puzzle

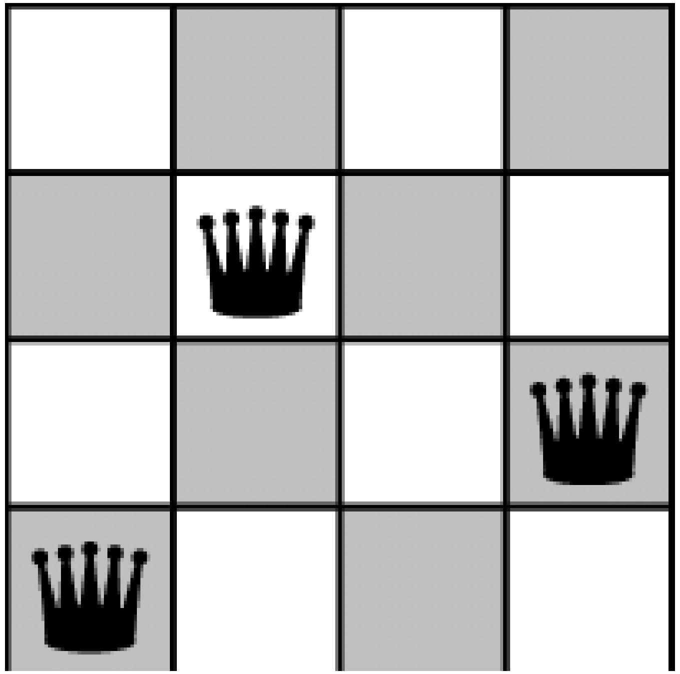
- One way to do this, is to define a function in this way

```
def ROW(row,dim):
    C=np.zeros((dim,dim), dtype=int)
    C[row,:] = 1
    return C.flatten()
```

```
R=ROW(0,4)
```

```
[[1 1 1 1]
 [0 0 0 0]
 [0 0 0 0]
 [0 0 0 0]]
```

```
[1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0]
```



```
N=4

w=1

B=np.zeros((N*N,N*N), dtype=float)
for row in range(N):
    R=ROW(row,N)
    for i in range(N*N):
        for j in range(i+1,N*N):
            B[i][j]=B[i][j]+R[i]*R[j]*w
print(B)
```

```
[[0. 1. 1. 1. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 1. 1. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 1. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 1. 1. 1. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 1. 1. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 1. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]]
```

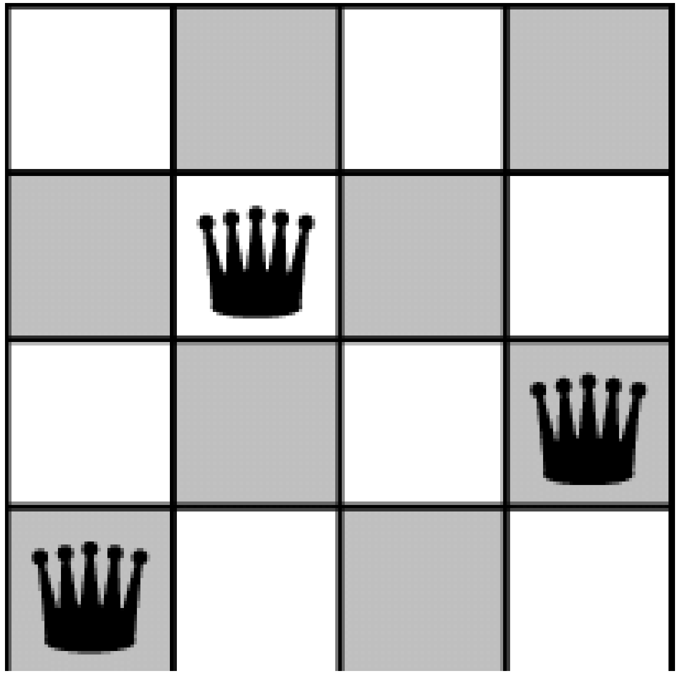
# Exercise 3: N-Queens Puzzle

- One way to do this, is to define a function in this way

```python
def ROW(row,dim):
    C=np.zeros((dim,dim), dtype=int)
    C[row,:] = 1
    return C.flatten()
```

```python
R=ROW(1,4)
```

```
[[0 0 0 0]
 [1 1 1 1]
 [0 0 0 0]
 [0 0 0 0]]


[0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0]
```



```python
N=4

w=1

B=np.zeros((N*N,N*N), dtype=float)
for row in range(N):
    R=ROW(row,N)
    for i in range(N*N):
        for j in range(i+1,N*N):
            B[i][j]=B[i][j]+R[i]*R[j]*w
print(B)
```

```
[[0. 1. 1. 1. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 1. 1. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 1. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 1. 1. 1. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 1. 1. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 1. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]]
```

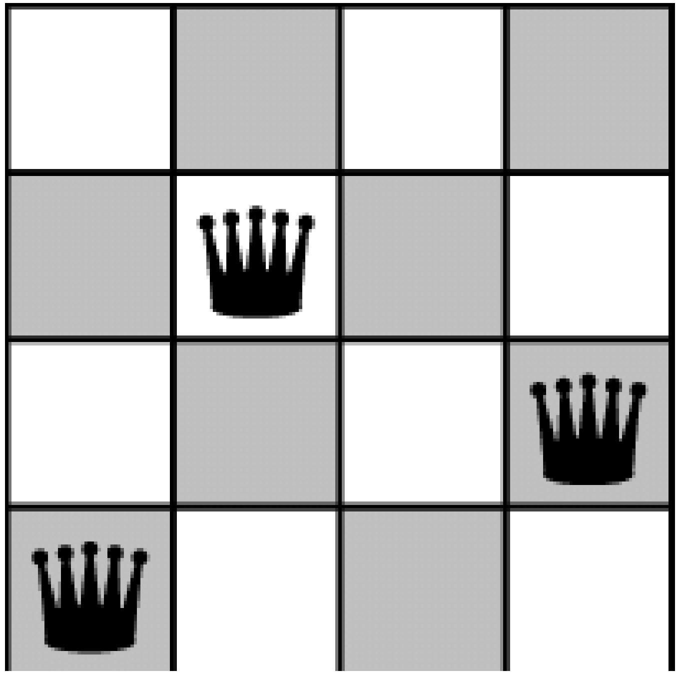
# Exercise 3: N-Queens Puzzle

- Same thing for columns

```python
def COL(col,dim):
    C=np.zeros((dim,dim), dtype=int)
    C[:,col] = 1
    return C.flatten()
```

```python
C=COL(0,4)
```

```
[[1 0 0 0]
 [1 0 0 0]
 [1 0 0 0]
 [1 0 0 0]]


[1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0]
```



```python
for col in range(N):
    C=COL(col,N)
    for i in range(N*N):
        for j in range(i+1,N*N):
            B[i][j]=B[i][j]+C[i]*C[j]*w

print(B)
```

```
[[0. 1. 1. 1. 1. 0. 0. 0. 1. 0. 0. 0. 1. 0. 0. 0.]
 [0. 0. 1. 1. 0. 1. 0. 0. 0. 1. 0. 0. 0. 1. 0. 0.]
 [0. 0. 0. 1. 0. 0. 1. 0. 0. 0. 1. 0. 0. 0. 1. 0.]
 [0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 1. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 1. 1. 1. 1. 0. 0. 0. 1. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 1. 1. 0. 1. 0. 0. 0. 1. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 1. 0. 0. 0. 1. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 1. 1. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 0. 1. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 1. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]]
```

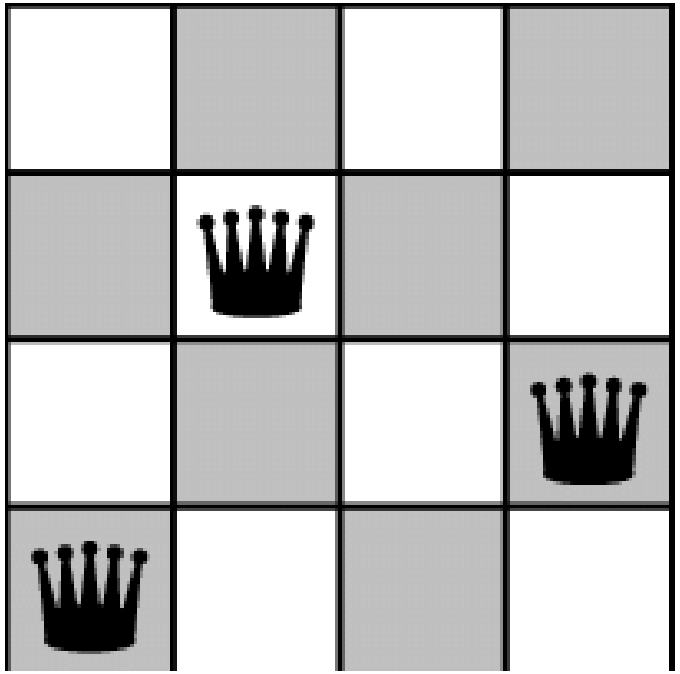
# Exercise 3: N-Queens Puzzle

- Same thing for columns

```python
def COL(col,dim):
    C=np.zeros((dim,dim), dtype=int)
    C[:,col] = 1
    return C.flatten()
```

```python
C=COL(1,4)
```

```
[[0 1 0 0]
 [0 1 0 0]
 [0 1 0 0]
 [0 1 0 0]]


[0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0]
```



```python
for col in range(N):
    C=COL(col,N)
    for i in range(N*N):
        for j in range(i+1,N*N):
            B[i][j]=B[i][j]+C[i]*C[j]*w

print(B)
```

```
[[0. 1. 1. 1. 1. 0. 0. 0. 1. 0. 0. 0. 1. 0. 0. 0.]
 [0. 0. 1. 1. 0. 1. 0. 0. 0. 1. 0. 0. 0. 1. 0. 0.]
 [0. 0. 0. 1. 0. 0. 1. 0. 0. 0. 1. 0. 0. 0. 1. 0.]
 [0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 1. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 1. 1. 1. 1. 0. 0. 0. 1. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 1. 1. 0. 1. 0. 0. 0. 1. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 1. 0. 0. 0. 1. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 1. 1. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 0. 1. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 1. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]]
```

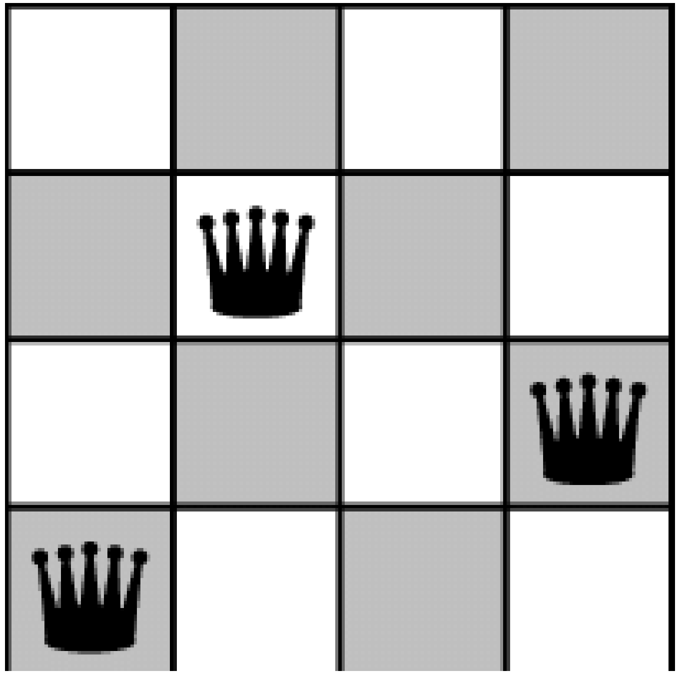
# Exercise 3: N-Queens Puzzle

- Same thing for diagonals

```python
def DIAG1(k,dim):
    d=np.ones(dim-abs(k), dtype=int)
    C=np.diag(d,k=k)
    return C.flatten()
```

```python
D=DIAG1(0,4)
```

```
[[1 0 0 0]
 [0 1 0 0]
 [0 0 1 0]
 [0 0 0 1]]


[1 0 0 0 0 1 0 0 0 0 1 0 0 0 0 1]
```



```python
for diag1 in range(-(N-2),N-1):
    D1=DIAG1(diag1,N)
    for i in range(N*N):
        for j in range(i+1,N*N):
            B[i][j]=B[i][j]+D1[i]*D1[j]*w

print(B)
```

```
[[0. 1. 1. 1. 1. 1. 0. 0. 1. 0. 1. 0. 1. 0. 0. 1.]
 [0. 0. 1. 1. 0. 1. 1. 0. 0. 1. 0. 1. 0. 1. 0. 0.]
 [0. 0. 0. 1. 0. 0. 1. 1. 0. 0. 1. 0. 0. 0. 1. 0.]
 [0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 1. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 1. 1. 1. 1. 1. 0. 0. 1. 0. 1. 0.]
 [0. 0. 0. 0. 0. 0. 1. 1. 0. 1. 1. 0. 0. 1. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 1. 1. 0. 0. 1. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 1. 1. 1. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 0. 1. 1. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]]
```

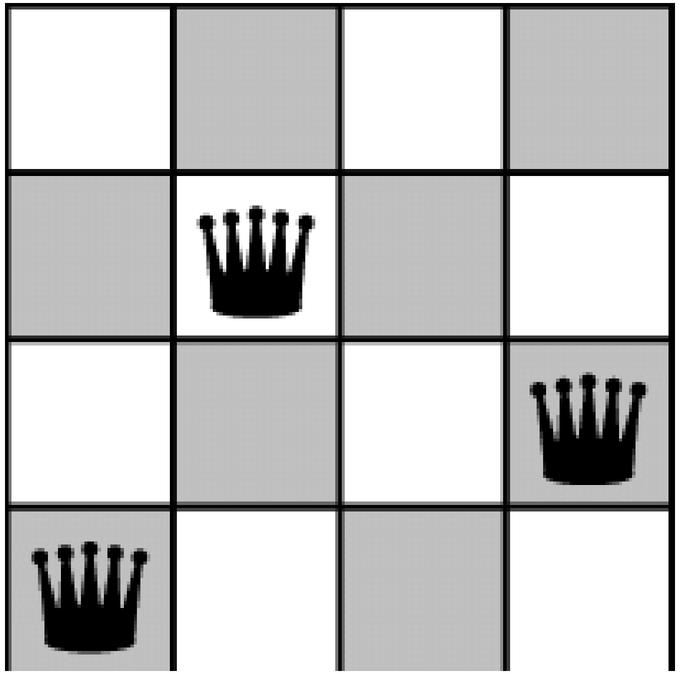
# Exercise 3: N-Queens Puzzle

- Same thing for diagonals

```python
def DIAG1(k,dim):
    d=np.ones(dim-abs(k), dtype=int)
    C=np.diag(d,k=k)
    return C.flatten()
```

```python
D=DIAG1(1,4)
```

```
[[0 1 0 0]
 [0 0 1 0]
 [0 0 0 1]
 [0 0 0 0]]


[0 1 0 0 0 0 1 0 0 0 0 1 0 0 0 0]
```



```python
for diag1 in range(-(N-2),N-1):
    D1=DIAG1(diag1,N)
    for i in range(N*N):
        for j in range(i+1,N*N):
            B[i][j]=B[i][j]+D1[i]*D1[j]*w

print(B)
```

```
[[0. 1. 1. 1. 1. 1. 0. 0. 1. 0. 1. 0. 1. 0. 0. 1.]
 [0. 0. 1. 1. 0. 1. 1. 0. 0. 1. 0. 1. 0. 1. 0. 0.]
 [0. 0. 0. 1. 0. 0. 1. 1. 0. 0. 1. 0. 0. 0. 1. 0.]
 [0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 1. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 1. 1. 1. 1. 1. 0. 0. 1. 0. 1. 0.]
 [0. 0. 0. 0. 0. 0. 1. 1. 0. 1. 1. 0. 0. 1. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 1. 1. 0. 0. 1. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 1. 1. 1. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 0. 1. 1. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]]
```

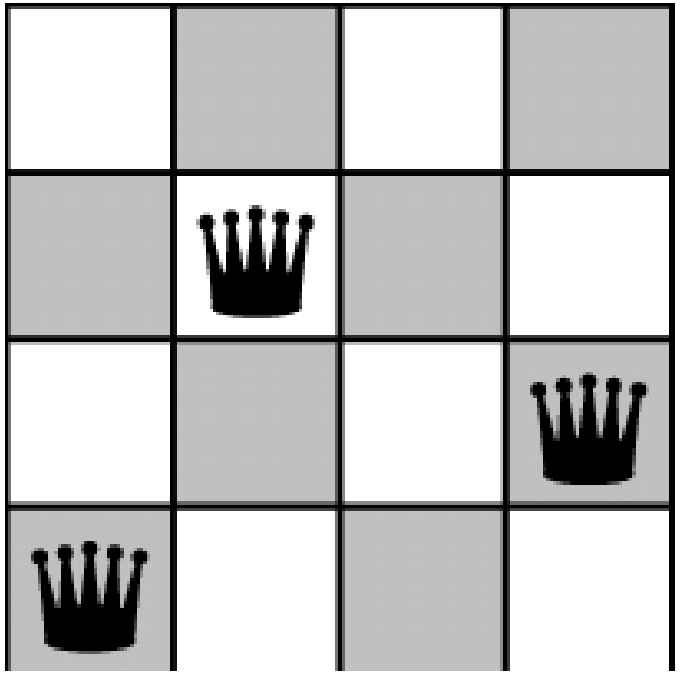
# Exercise 3: N-Queens Puzzle

- Same thing for diagonals

```python
def DIAG1(k,dim):
    d=np.ones(dim-abs(k), dtype=int)
    C=np.diag(d,k=k)
    return C.flatten()
```

```python
D=DIAG1(-1,4)
```

```
[[0 0 0 0]
 [1 0 0 0]
 [0 1 0 0]
 [0 0 1 0]]


[0 0 0 0 1 0 0 0 0 1 0 0 0 0 1 0]
```



```python
for diag1 in range(-(N-2),N-1):
    D1=DIAG1(diag1,N)
    for i in range(N*N):
        for j in range(i+1,N*N):
            B[i][j]=B[i][j]+D1[i]*D1[j]*w

print(B)
```

```
[[0. 1. 1. 1. 1. 1. 0. 0. 1. 0. 1. 0. 1. 0. 0. 1.]
 [0. 0. 1. 1. 0. 1. 1. 0. 0. 1. 0. 1. 0. 1. 0. 0.]
 [0. 0. 0. 1. 0. 0. 1. 1. 0. 0. 1. 0. 0. 0. 1. 0.]
 [0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 1. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 1. 1. 1. 1. 1. 0. 0. 1. 0. 1. 0.]
 [0. 0. 0. 0. 0. 0. 1. 1. 0. 1. 1. 0. 0. 1. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 1. 1. 0. 0. 1. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 1. 1. 1. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 0. 1. 1. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]]
```

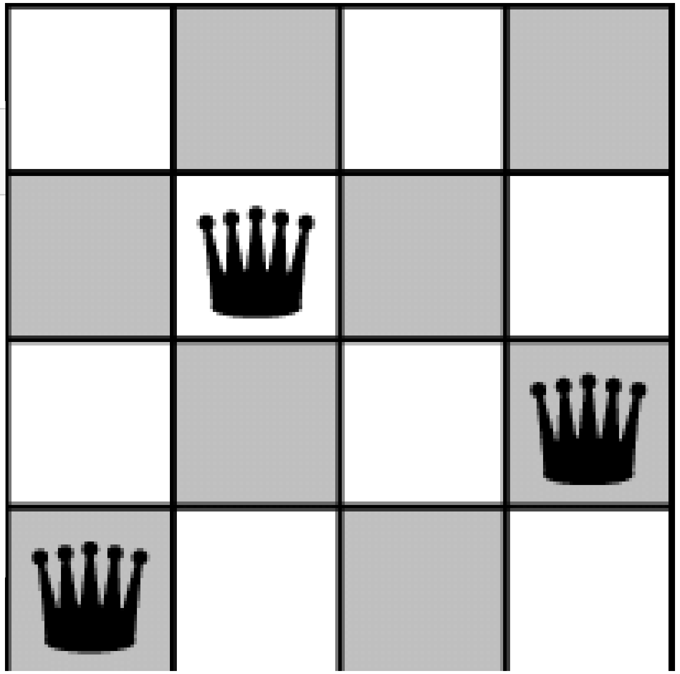
# Exercise 3: N-Queens Puzzle

- Same thing for diagonals

```python
def DIAG2(k,dim):
    d=np.ones(dim-abs(k), dtype=int)
    C=np.fliplr(np.diag(d,k=k))
    return C.flatten()
```

```python
D=DIAG2(0,4)
```

```
[[0 0 0 1]
 [0 0 1 0]
 [0 1 0 0]
 [1 0 0 0]]


[0 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0]
```



```python
for diag2 in range(-(N-2),N-1):
    D2=DIAG2(diag2,N)
    for i in range(N*N):
        for j in range(i+1,N*N):
            B[i][j]=B[i][j]+D2[i]*D2[j]*w

print(B)
```

```
[[0. 1. 1. 1. 1. 1. 0. 0. 1. 0. 1. 0. 1. 0. 0. 1.]
 [0. 0. 1. 1. 1. 1. 1. 0. 0. 1. 0. 1. 0. 1. 0. 0.]
 [0. 0. 0. 1. 0. 1. 1. 1. 1. 0. 1. 0. 0. 0. 1. 0.]
 [0. 0. 0. 0. 0. 0. 1. 1. 0. 1. 0. 1. 1. 0. 0. 1.]
 [0. 0. 0. 0. 0. 1. 1. 1. 1. 1. 0. 0. 1. 0. 1. 0.]
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 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 0. 1. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 1. 1. 1. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 1. 1. 1. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 1. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]]
```

SuperComputing Applications and Innovation

**Quantum Annealing with continuous variables:**
**Low-Rank Matrix Factorization**

Daniele Ottaviani
CINECA

**Quantum Computing and High Performance Computing**
CINECA Casalecchio di Reno, Bologna, 18-12-2018

# QUBO Problems with real variables

**We define a QUBO problem with real variables as a Quadratic Unconstrained Optimization problem with unknown variables expressed as:**

$$x = c \cdot \sum_{e=0}^{N-1} 2^e q_e, \quad c = 10^{-a}, \text{ for some } a \in \mathbb{N}$$

For example, the QUBO problem associated with the simple equation $x - b = 0$ is:

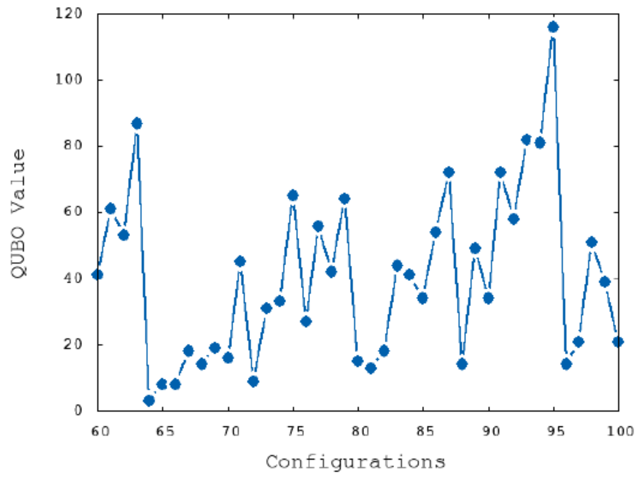
$$\min_{\mathbf{q}=(q_0,\ldots,q_{N-1})} \left( \sum_{e=0}^{N-1} \left(c^2 2^{2e} - bc2^{e+1}\right) q_e + \sum_{e<f} \left(c^2 2^{e+f+1}\right) q_e q_f \right)$$

Considering $x - b = 0$ as $\min_{x \in \mathbb{R}} (x - b)^2$

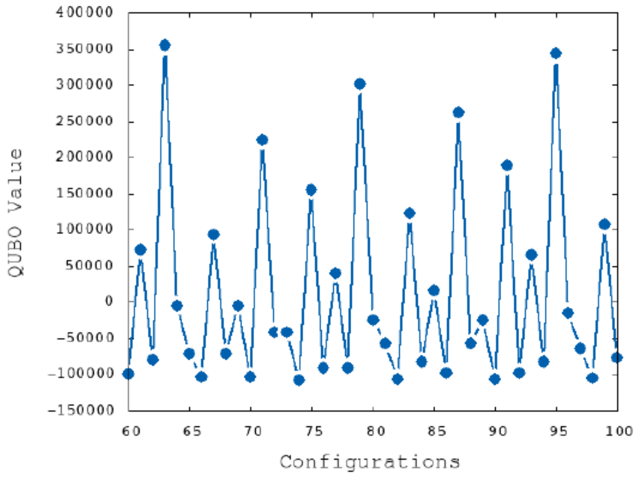
# Graphical representation

**QUBO problems of this kind are particularly difficult to solve.**
**Especially with annealing techniques.**

**This is due to the exponential dependence of the coefficients from the binary variable indices, which create numerous local minima very similar to the global minimum.**




**"Normal" QUBO landscape**




**"Real-variables" QUBO landscape**

# Solving a linear system

We have chosen to solve a linear system $A\mathbf{x} = b$, where

$\mathbf{x} = (x_1, x_2, x_3)$ and $x_i \in [0,1]$.

We represent $x_i = c \cdot \sum_{e=0}^{9} 2^e q_e, \quad c = 10^{-3}$ $(N = 10,\ a = 3)$.

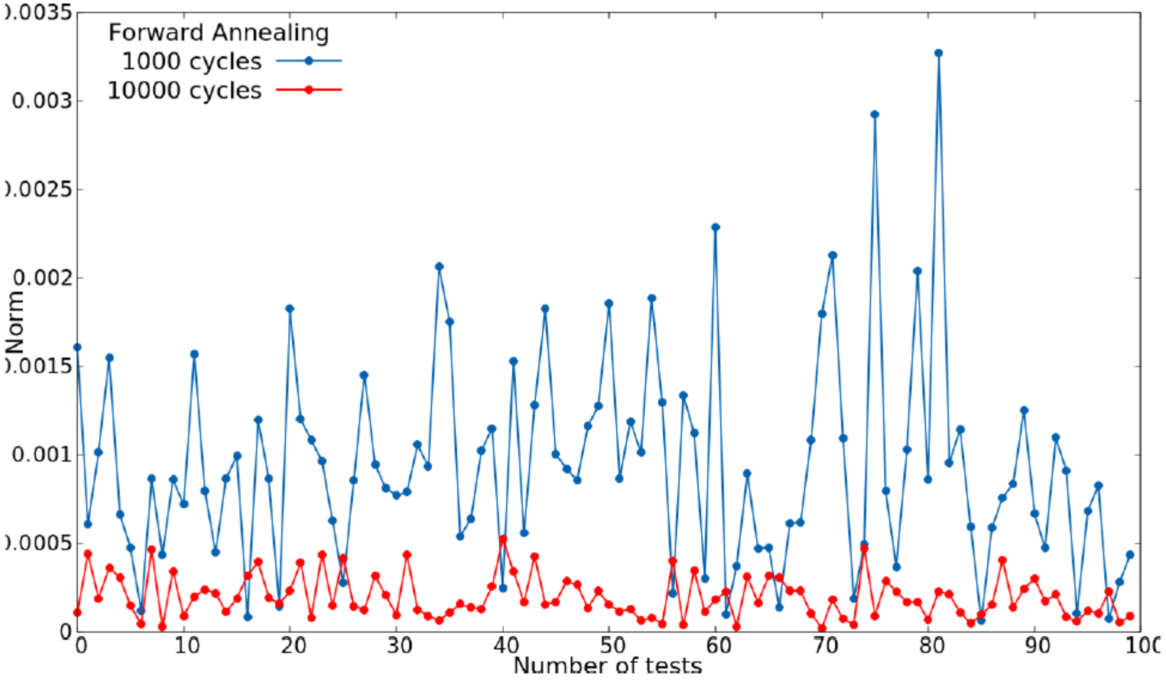
We will find $\mathbf{x}$ solving $\min_{\mathbf{x}\in[0,1]^3} ||A\mathbf{x} - b||_2^2$

$$\begin{bmatrix} 1.301 & 0.125 & 0.187 \\ 0.440 & 0.342 & 0.082 \\ 0.672 & 0.709 & 0.802 \\ 0.218 & 0.427 & 0.520 \\ 0.024 & 0.036 & 0.038 \end{bmatrix} \cdot \begin{bmatrix} 0.178 \\ 0.333 \\ 0.489 \end{bmatrix} = \begin{bmatrix} 0.365 \\ 0.232 \\ 0.748 \\ 0.435 \\ 0.035 \end{bmatrix}$$

# Advanced Annealing Techiques

## Attempt number 1: Forward Annealing

100 attempts with 1,000 and 10,000 annealing cycles




$$\begin{bmatrix} 0.179 \\ 0.338 \\ 0.483 \end{bmatrix}$$

$$\left(\begin{bmatrix} 0.178 \\ 0.333 \\ 0.489 \end{bmatrix}\right)$$

# Local refinement of solutions:

## Reverse Annealing

**Introduced with the last D-Wave model, DWAVE2000Q**

**Starting point chosen by the user**

**Backward Annealing**

**Forward Annealing**

**During the Backward Annealing phase, the transverse field slowly increases up to a value chosen by the user (*Reversal Distance)***

**The last Forward Annealing phase is a LOCAL quantum annealing search: how much local depends on the reversal distance value.**

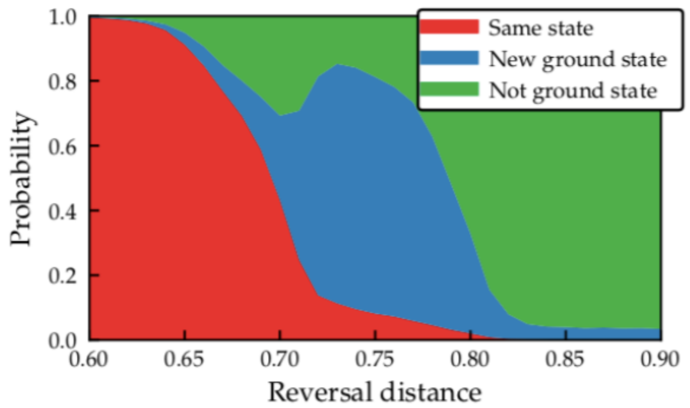
Image taken from *Reverse Quantum Annealing for Local Refinement of Solutions,* D-Wave White Papers, 2017

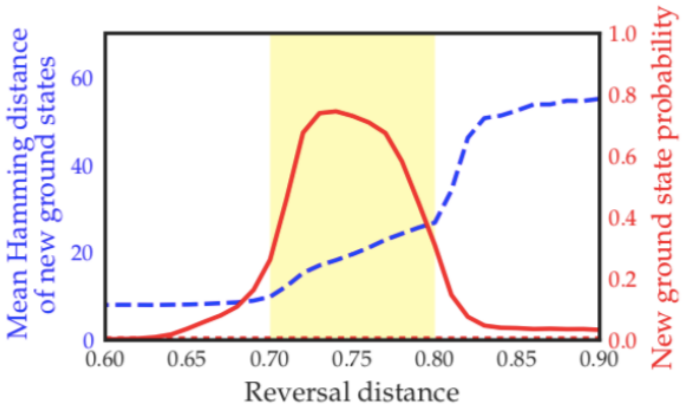
# Tuning the reversal distance

D-WAVE
The Quantum Computing Company™

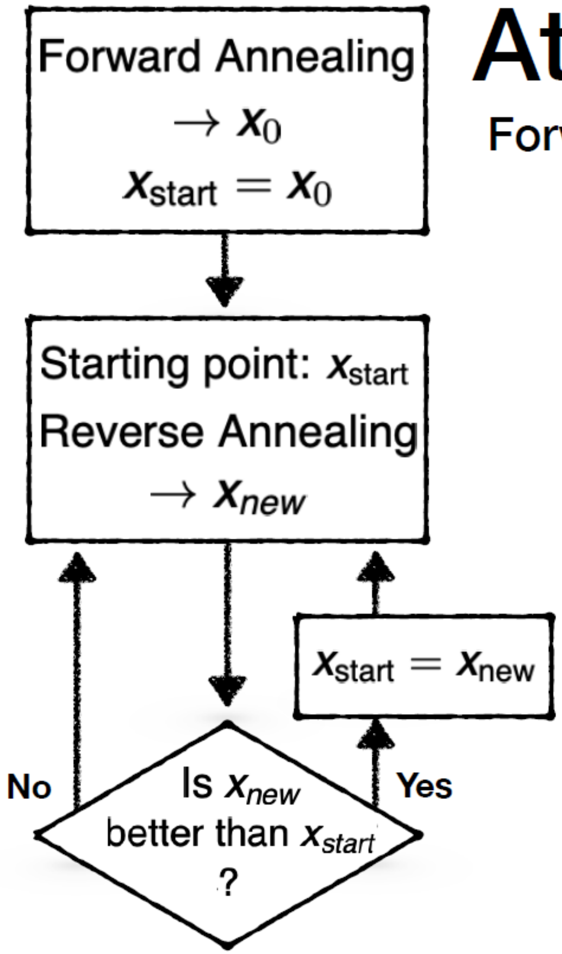
Reverse Quantum Annealing for Local Refinement of Solutions

WHITEPAPER



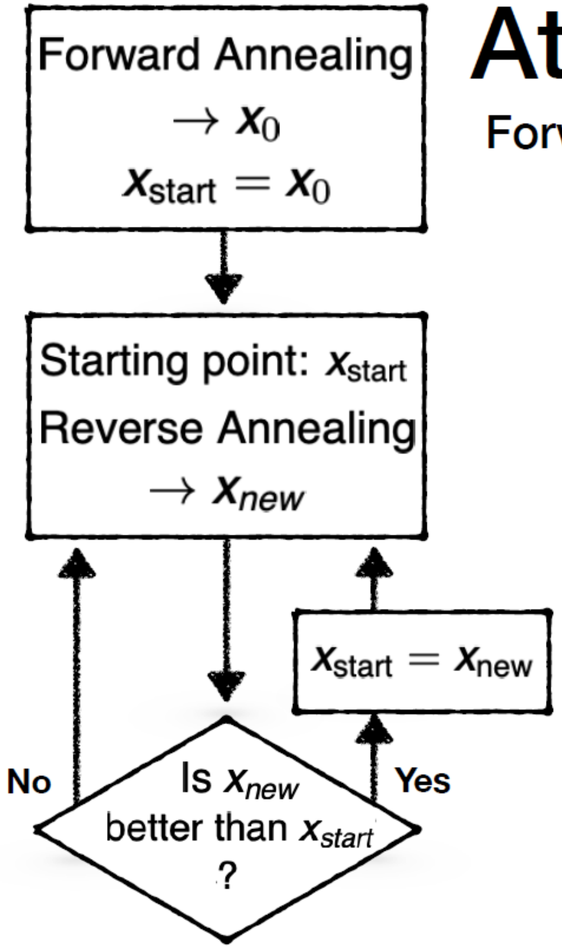
# Advanced Annealing Techiques




## Attempt number 2:

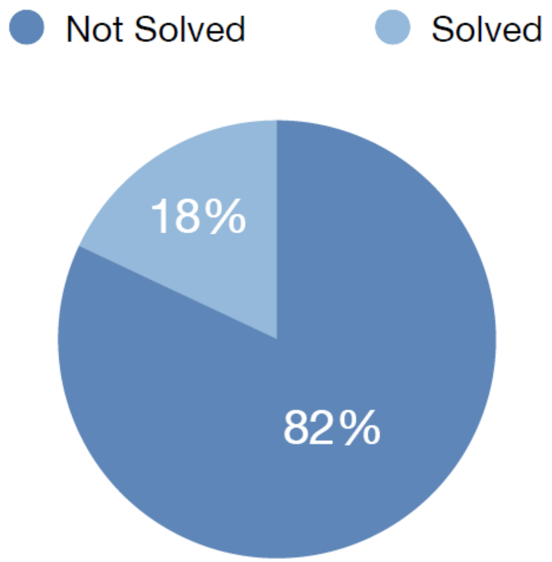
Forward Annealing + Reverse Annealing

# Advanced Annealing Techiques




## Attempt number 2:

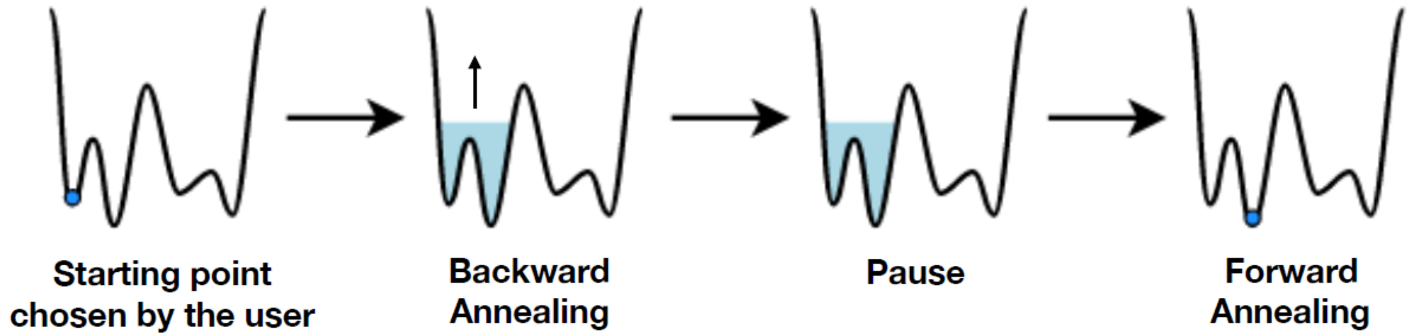
Forward Annealing + Reverse Annealing



# Pausing the annealing process

Being able to pause the annealing process is another of the new features introduced with the latest D-WAVE quantum annealer.

We can use the pause during a Reverse Annealing search in this way:




Why pause? Because pausing the annealing process means better exploration of the selected zone, increasing the chances of obtaining a new global minimum.

But pay attention: pause can't be too long. For two main reasons:
1) it increase the computational time of each annealing cycle.
2) if it is too long, it may also risk to increase the search radius more than desired.

## Correlation between pause and search radius

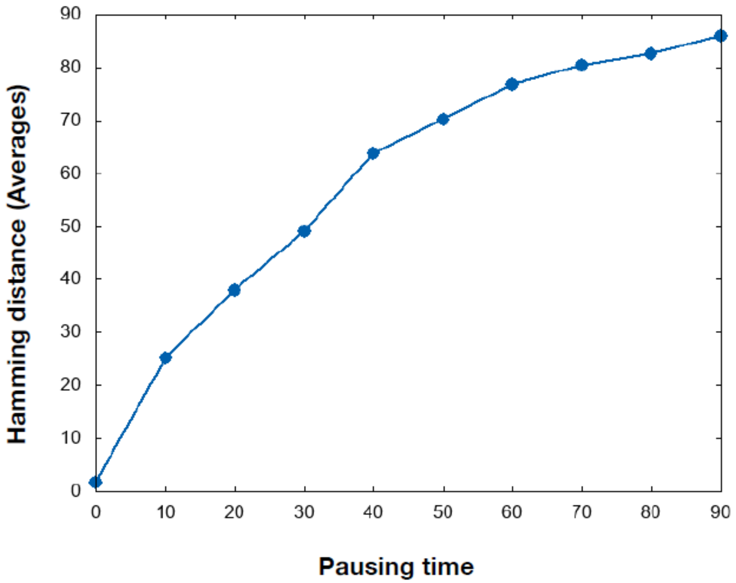
**We can realize a posteriori the search radius of a reverse annealing search by analyzing the average distance between the solutions found by each cycle.**

**To do this, we choose the Hamming distance, a function written to calculate the distance between vectors of binary numbers.**

**We have observed that there is a correlation between the pause time and the average distance between the solutions obtained with each annealing cycle**

**As with the reversal distance, here too we have to be careful about the right break time:**

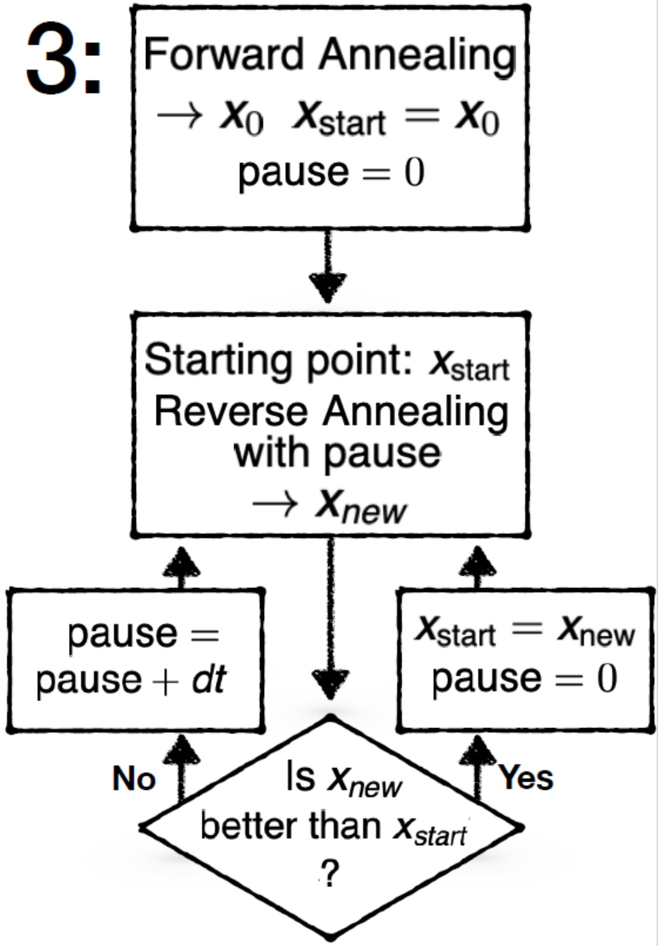
**too little is not enough,**
**too much can lead to wrong results**



# Advanced Annealing Techiques

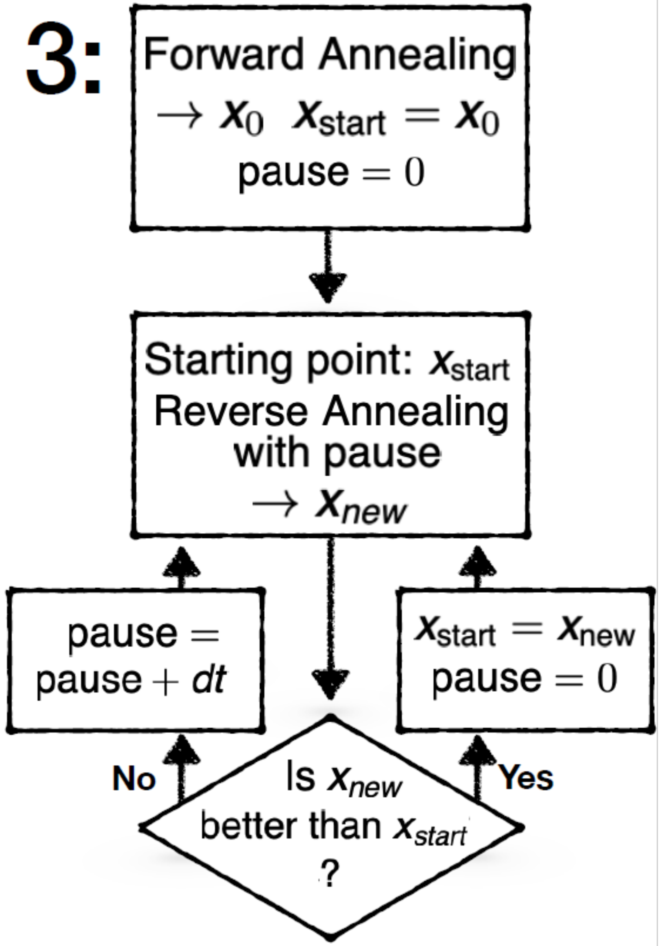
## Attempt number 3:

Forward Annealing +
Reverse Annealing with pause



# Attempt number 3:

Forward Annealing +
Reverse Annealing with pause



# Advanced Annealing Techiques

## Attempt number 3:

**Forward Annealing +**
**Reverse Annealing with pause**