Introduction to Quantum Computing
Day 2 - Quantum Error Correction

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Intro to QEC
Common sources of errors in QC

- **Coherent quantum errors**: Gates which are incorrectly applied
- **Environmental decoherence**: errors due to the interaction with the external environment
- **Initialization errors**: failing to prepare the correct initial state
- **Qubit loss**
Classical error correction employs redundancy.

The simplest way is to store the information multiple times, and just take a majority vote if these copies are later found to disagree.

\[
0 \rightarrow 0000
\]
\[
1 \rightarrow 1111
\]
Quantum Error Correction

It is possible to reuse redundancy in quantum error correction. However, there are some complications:

- **No-cloning Theorem**
- Qubits are susceptible to both bit-flips (X-errors) and phase-flips (Z-errors). (Classically, only bit-flip errors)
- Measuring affects the quantum state. Detecting errors must not compromise encoded information
Intro to QEC

**Cryptography**
- Shor's Algorithm
  - Exponential Speedup
- RSA

**Optimization**
- Grover's Algorithm
  - Quadratic Speedup
Intro to QEC

Cryptography
Shor’s Algorithm
Exponential Speedup

Optimization
Grover’s Algorithm
Quadratic Speedup

These algorithms assume to have ideal qubits that are not subjected to noise and errors.
Intro to QEC

Cryptography
Shor’s Algorithm
Exponential Speedup

Optimization
Grover’s Algorithm
Quadratic Speedup

• Require error corrected (fault-tolerant) quantum computers with about 1 million or 100 thousands of qubits

• Will be available in 10-20 years
Quantum Error Correction

It is possible to reuse redundancy in quantum error correction. However, there are some complications:

• No-cloning Theorem

• Qubits are susceptible to both bit-flips (X-errors) and phase-flips (Z-errors). (Classically, only bit-flip errors)

• Measuring affects the quantum state. Detecting errors must not compromise encoded information
No Cloning Theorem
No Cloning Theorem

No Cloning theorem

It does NOT exist an universal cloning machine which is a unitary transformation such that

\[ U |\psi\rangle |\chi\rangle = |\psi\rangle |\psi\rangle \]

\[ \forall |\psi\rangle \in \mathcal{H} \quad \text{and} \quad |\chi\rangle \in \mathcal{H} \quad \text{fixed} \]
Suppose such universal cloning machine exists and apply it to two states like below:

\[ U |\psi_1 \rangle |\psi_2 \rangle = |\psi_1 \rangle |\psi_1 \rangle \quad U |\psi_2 \rangle |\psi_2 \rangle = |\psi_2 \rangle |\psi_2 \rangle \]
Suppose such universal cloning machine exists and apply it to two states like below:

\[ U |\psi_1\rangle |\alpha\rangle = |\psi_1\rangle |\psi_1\rangle \quad U |\psi_2\rangle |\alpha\rangle = |\psi_2\rangle |\psi_2\rangle \]

Consider a scalar product between the terms of the eqn.s above:

\[ \langle \alpha | \langle \psi_2 | U^* U |\psi_1\rangle |\alpha\rangle = \langle \psi_2 | \langle \psi_2 | \psi_1\rangle |\psi_1\rangle \]
No Cloning Theorem

Suppose such universal cloning machine exists and apply it to two states like below

\[ U |\Psi_1\rangle |\alpha\rangle = |\Psi_1\rangle |\Psi_1\rangle \]
\[ U |\Psi_2\rangle |\alpha\rangle = |\Psi_2\rangle |\Psi_2\rangle \]

Consider a scalar product between the terms of the eqn.s above

\[ \langle \alpha | \langle \Psi_2 | U^\dagger U |\Psi_1\rangle |\alpha\rangle = \langle \Psi_2 | \langle \Psi_2 | \Psi_1 \rangle |\Psi_1\rangle \]
\[ \langle \alpha | \alpha \rangle \langle \Psi_2 | \Psi_1 \rangle = | \langle \Psi_2 | \Psi_1 \rangle |^2 \]
\[ \langle \Psi_2 | \Psi_1 \rangle = | \langle \Psi_2 | \Psi_1 \rangle |^2 \]
Suppose such universal cloning machine exists and apply it to two states like below

\[ U |\Psi_1\rangle |d\rangle = |\Psi_1\rangle |\Psi_1\rangle \quad U |\Psi_2\rangle |d\rangle = |\Psi_2\rangle |\Psi_2\rangle \]

Consider a scalar product between the terms of the eqn.s above

\[ \langle d | \langle \Psi_2 | U^\dagger U |\Psi_1\rangle |d\rangle = \langle \Psi_2 | \langle \Psi_2 | \Psi_1 \rangle |\Psi_1\rangle \]

\[ \langle d | d \rangle \langle \Psi_2 | \Psi_1 \rangle = | \langle \Psi_2 | \Psi_1 \rangle |^2 \]

\[ \langle \Psi_2 | \Psi_1 \rangle = \sqrt{\langle \Psi_2 | \Psi_1 \rangle} \]

Contradiction (true only for orthogonal states)

\[ \begin{cases} \langle \Psi_2 | \Psi_2 \rangle = 0 \\ \langle \Psi_1 | \Psi_2 \rangle = 1 \end{cases} \]
QEC:
Three qubits repetition codes
Three qubits repetition codes

Repetition codes

This techniques use redundancy, entanglement and syndrome measurements to correct single qubits bit-flip and phase-flip errors which may occur with some probability $p$.

\[
U_{\text{err}}^X = (1 - p) \mathbb{I} + p X
\]

\[
U_{\text{err}}^Z = (1 - p) \mathbb{I} + p Z
\]
Three qubits repetition codes

Bit-flip error code

Encoding

\[ |0\rangle \rightarrow |0\ 0\ 0\rangle \]

\[ |1\rangle \rightarrow |1\ 1\ 1\rangle \]

Syndrome Measurement

\[ (Z_1 Z_2), (Z_2 Z_3) \]
Three qubits repetition codes

Bit-flip errors code
Bit-flip error codes

Step by step analysis (two qubit case)

**Initial state**

\[ \lvert \Psi \rangle = \alpha \lvert 0 \rangle + \beta \lvert 1 \rangle \]

\[ \lvert \Psi \rangle \rightarrow 10 \]

\[ \lvert \Psi \rangle \rightarrow U_{\text{ERR}} \]

\[ \lvert \Psi \rangle \rightarrow Z_2 Z_2 \]

\[ \lvert \Psi \rangle \rightarrow H \]

\[ \lvert \Psi \rangle \rightarrow \lnot \]

Diagram with quantum gates and state transitions.
Bit-flip error codes

Step by step analysis (two qubit case)

**Encoding**

\[ |0\rangle \rightarrow |0_L\rangle = |00\rangle \]
\[ |1\rangle \rightarrow |1_L\rangle = |11\rangle \]

Obtained with a Control-X

\[ U_{C_X} |\psi\rangle |0\rangle = a|00\rangle + b|11\rangle \]
\[ = a|0_L\rangle + b|1_L\rangle \]
Bit-flip error codes

Step by step analysis (two qubit case)

Assume a bit-flip error on the first qubit

\[
U_{\text{ERR}} = X_1
\]

\[
|00\rangle + \beta |11\rangle \xrightarrow{U_{\text{ERR}}} \alpha |10\rangle + \beta |01\rangle
\]
Bit-flip error codes

Step by step analysis (two qubit case)

Assume a bit-flip error on the first qubit

\[
U_{\text{ERR}} = X_1
\]

\[
|00\rangle + \beta |11\rangle \xrightarrow{U_{\text{ERR}}} |10\rangle + \beta |01\rangle
\]

\[
|0\rangle_A \xrightarrow{H} \left( \frac{|0\rangle_A + 1|1\rangle_A}{\sqrt{2}} \right)
\]
Bit-flip error codes

Step by step analysis (two qubit case)

Assume a bit-flip error on the first qubit

\[ U_{ERR} = X_1 \]

\[ \alpha |00\rangle + \beta |11\rangle \xrightarrow{U_{ERR}} \alpha |10\rangle + \beta |01\rangle \]

\[ |0\rangle_A \xrightarrow{H} \left( \frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}} \right) \]

\[ (\alpha |10\rangle + \beta |01\rangle) \left( \frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}} \right) \]
Bit-flip error codes

Step by step analysis (two qubit case)

Control gate for syndrome measurement

\[
\begin{align*}
(\alpha|0\rangle + \beta|1\rangle) \left(\frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}}\right) \\
|1\rangle_A \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) - |1\rangle_A \left(\frac{2|0\rangle + \beta|1\rangle}{\sqrt{2}}\right)
\end{align*}
\]
Bit-flip error codes

Step by step analysis (two qubit case)

Control gate for syndrome measurement

\[
\begin{align*}
\langle \psi | & 10 \rangle + \beta | 01 \rangle \left( \frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}} \right) \\
= & \left( |0\rangle_A \left( \frac{a |0\rangle + \beta |1\rangle}{\sqrt{2}} \right) - |1\rangle_A \left( \frac{a |0\rangle + \beta |1\rangle}{\sqrt{2}} \right) \right) \left( \frac{|0\rangle_A - |1\rangle_A}{\sqrt{2}} \right)
\end{align*}
\]
Bit-flip error codes

Step by step analysis (two qubit case)

\[\alpha|10\rangle + \beta|01\rangle \rightarrow \left( \frac{|10\rangle_A - |11\rangle_A}{\sqrt{2}} \right) \rightarrow H_A \rightarrow \left( \alpha|10\rangle + \beta|01\rangle \right) |\pm\rangle_A \]
Bit-flip error codes

Step by step analysis (two qubit case)

Measuring the state of the ancillary qubit in $|\tilde{z}\rangle_A$ will reveal that a bit-flip error has occurred.
Bit-flip error codes

Step by step analysis (two qubit case)

\[ \alpha |0\rangle + \beta |1\rangle \left( \frac{|0\rangle_A - |1\rangle_A}{\sqrt{2}} \right) \xrightarrow{H_A} \left( \alpha |0\rangle + \beta |1\rangle \right) |\pm\rangle_A \]

Measuring the state of the ancillary qubit in \( |\pm\rangle_A \) will reveal that a bit-flip error has occurred.

We need 3 qubits in the encoding to perfectly locate where bit-flip error has occurred.
Bit-flip error codes

Step by step analysis (two qubit case)

<table>
<thead>
<tr>
<th>Outcome</th>
<th>$\bar{z}_1 \bar{z}_2$</th>
<th>$z_2 \bar{z}_3$</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>+1</td>
<td>+1</td>
<td></td>
<td>None</td>
</tr>
<tr>
<td>-1</td>
<td>+1</td>
<td></td>
<td>First</td>
</tr>
<tr>
<td>+1</td>
<td>-1</td>
<td></td>
<td>Third</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td></td>
<td>Second</td>
</tr>
</tbody>
</table>

We need 3 qubits in the encoding to perfectly locate where bit-flip error has occurred.
Bit-flip error codes

Step by step analysis (two qubit case)

Assume a no error on the first qubit

\[ U_{\text{ERR}} = I \]

\[ \alpha |00\rangle + \beta |11\rangle \xrightarrow{U_{\text{ERR}}} \alpha |00\rangle + \beta |11\rangle \]

\[ (\alpha |00\rangle + \beta |11\rangle) \left( \frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}} \right) \xrightarrow{C-Z_1Z_2} \left( \frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}} \right) \]

\[ \xrightarrow{H_A} \left( \alpha |00\rangle + \beta |11\rangle \right) |0\rangle_A \]

Measuring the state of the ancillary qubit in 0 reveals that no error has occurred
Three qubits repetition codes

Phase-flip errors code

**Encoding**

\[
|0\rangle \rightarrow |+++\rangle \\
|1\rangle \rightarrow |-\rangle
\]

\[
|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \\
|-\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}
\]

**Syndrome Measurement**

\[
(X_1X_2), (X_2X_3)
\]
Three qubits repetition codes

Phase-flip errors code

\[ |\psi\rangle, |0\rangle, |0\rangle \]

\[ U^2_{\text{ERR}} \]

\[ X_1 X_2, X_2 X_3 \]

\[ |0\rangle_{A_1}, |0\rangle_{A_2} \]
Phase-flip error codes

Step by step analysis (two qubit case)

Initial state

\[ |\psi\rangle = \alpha |0\rangle + \beta |1\rangle \]

\[ |\psi\rangle |0\rangle \]
Phase-flip error codes

Step by step analysis (two qubit case)

Encoding

\[ |0\rangle \rightarrow |+ +\rangle \]
\[ |1\rangle \rightarrow | - -\rangle \]

Obtained with a Control-X and Hadamards

\[
\left(\alpha |0\rangle + \beta |1\rangle\right) |0\rangle \xrightarrow{U_{\text{err}}} \alpha |00\rangle + \beta |11\rangle \xrightarrow{H^\otimes 2} \alpha \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) + \beta \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = \\
= \alpha | + +\rangle + \beta | - -\rangle
\]
Phase-flip error codes

Step by step analysis (two qubit case)

Assume a phase-flip error on the first qubit

\[
U_{\text{ERR}} = Z_1
\]

\[
\alpha |+\rangle + \beta |-\rangle \xrightarrow{U_{\text{ERR}}} \alpha |-\rangle + \beta |+\rangle
\]

\[
|\psi\rangle \xrightarrow{H} H|\psi\rangle \xrightarrow{U_{\text{ERR}}} H|\psi\rangle \xrightarrow{Z_1} H|\psi\rangle \\
10\rangle \xrightarrow{H} H10\rangle \xrightarrow{X_1X_2} H10\rangle
\]
Phase-flip error codes

Step by step analysis (two qubit case)

Assume a phase-flip error on the first qubit

\[
U_{\text{EE}} = Z_1
\]

\[
|\psi\rangle = |+\rangle + |\rangle \rightarrow U_{\text{EE}} |\psi\rangle = |+\rangle + Z_1 |\rangle = |+\rangle + Z_1 |\rangle
\]

\[
|\psi\rangle = |+\rangle + Z_1 |\rangle
\]

\[
|\psi\rangle = \frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}}
\]

CINECA
Phase-flip error codes

Step by step analysis (two qubit case)

Assume a phase-flip error on the first qubit

\[
\begin{bmatrix}
U_{\text{ERR}} = \mathbb{Z}_1
\end{bmatrix}
\]

\[
\alpha |+\rangle + \beta |-\rangle \xrightarrow{U_{\text{ERR}}} \alpha |-\rangle + \beta |+\rangle
\]

\[
|0\rangle_A \xrightarrow{H} \left( \frac{|0\rangle_A + |\pm\rangle_A}{\sqrt{2}} \right)
\]

\[
(\alpha |-\rangle + \beta |+\rangle) \left( \frac{|0\rangle_A + |\pm\rangle_A}{\sqrt{2}} \right)
\]
Phase-flip error codes

Step by step analysis (two qubit case)

Control gate for syndrome measurement

\[
\left(\alpha \left|+\right> + \beta \left|-\right>\right)\left(\frac{\left|0\right>_A + \left|1\right>_A}{\sqrt{2}}\right)
\]

\[
\frac{1}{\sqrt{2}} \left|X\right>_A \left|+\right>_A = \left|+\right>_A
\]

\[
\frac{1}{\sqrt{2}} \left|X\right>_A \left|-\right>_A = \left|-\right>_A
\]

\[
\left|0\right>_A \left(\alpha \left|+\right> + \beta \left|-\right>\right) - \left|1\right>_A \left(\alpha \left|+\right> + \beta \left|-\right>\right) = (\alpha \left|+\right> + \beta \left|-\right>) \left(\frac{\left|0\right>_A - \left|1\right>_A}{\sqrt{2}}\right)
\]
Phase-flip error codes

Step by step analysis (two qubit case)

\[
\left( \alpha |+\rangle + \beta |-\rangle \right) \left( \frac{10^A_A - 12^A_A}{\sqrt{2}} \right)
\]

\[
\left( \alpha |+\rangle + \beta |-\rangle \right) |2\rangle_A
\]
Phase-flip error codes

Step by step analysis (two qubit case)

\[ (\alpha | + \rangle + \beta | - \rangle) \left( \frac{1}{\sqrt{2}} (|0 \rangle_A - i |1 \rangle_A) \right) \]

Measuring the state of the ancillary qubit in \(|z\rangle_A\) will reveal that a phase-flip error has occurred.
Step by step analysis (two qubit case)

Measuring the state of the ancillary qubit in $|z\rangle_A$ will reveal that a phase-flip error has occurred.

We need 3 qubits in the encoding to perfectly locate where phase-flip error has occurred.
# Phase-flip error codes

## Step by step analysis (two qubit case)

We need 3 qubits in the encoding to perfectly locate where phase-flip error has occurred.

<table>
<thead>
<tr>
<th>Outcome</th>
<th>$(X_1 X_2)$</th>
<th>$(X_2 X_3)$</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>+1</td>
<td>+1</td>
<td>+1</td>
<td>None qubits</td>
</tr>
<tr>
<td>-1</td>
<td>+1</td>
<td>+1</td>
<td>First qubit</td>
</tr>
<tr>
<td>+1</td>
<td>-1</td>
<td>-1</td>
<td>Third qubit</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>Second qubit</td>
</tr>
</tbody>
</table>
Shor Code
Shor code

Nine qubit repetition code
Uses nine qubits to correct both bit-flip and phase-flip errors

Encoding

Encoding 1
\[ |0\rangle \rightarrow |+++\rangle \quad \text{then} \quad |1\rangle \rightarrow |---\rangle \]

Encoding 2
\[ |0\rangle \rightarrow |000\rangle \quad |1\rangle \rightarrow |111\rangle \]
Shor code

Nine qubit repetition code
Uses nine qubits to correct both bit-flip and phase-flip errors

Encoding

Encoding 1
\[ |0\rangle \rightarrow \left| + + + \right\rangle \quad \text{then} \quad |1\rangle \rightarrow \left| - - - \right\rangle \]

Encoding 2
\[ |0\rangle \rightarrow \left| 0 0 0 \right\rangle \quad \quad |1\rangle \rightarrow \left| 1 1 1 \right\rangle \]
Shor code

Nine qubit repetition code
Uses nine qubits to correct both bit-flip and phase-flip errors

<table>
<thead>
<tr>
<th>Encoding 1</th>
<th>Encoding 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>0\rangle \rightarrow</td>
</tr><tr>
<td>angle$</td>
<td>$</td>
</tr>
<tr>
<td>$</td>
<td>1\rangle \rightarrow</td>
</tr>
</tbody>
</table>

Syndrome Measurement

$\begin{pmatrix}
X_1 X_2 X_3 \\
X_4 X_5 X_6 \\
X_7 X_8 X_9
\end{pmatrix}
\begin{pmatrix}
Z_1 Z_2 Z_3 \\
Z_4 Z_5 \\
Z_7 Z_8 Z_9
\end{pmatrix}
$
Quantum Computing @ CINECA

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- Internship programs, Courses and Conference (HPCQC)

https://www.quantumcomputinglab.cineca.it

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