Introduction to Quantum Computing
Day 2 - Quantum Error Correction

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22 June 2021

CINECA QUANTUM COMPUTING LAB

Intro to QEC

Common sources of errors in QC

- **Coherent quantum errors**: Gates which are incorrectly applied
- **Environmental decoherence**: errors due to the interaction with the external environment
- **Initialization errors**: failing to prepare the correct initial state
- **Qubit loss**

Classical Error Correction

Classical error correction employs **redundancy**.

The simplest way is **to store the information multiple times**, and just take a **majority vote** if these copies are later found to disagree

$$0 \longrightarrow 0\ 0\ 0$$

$$1 \longrightarrow 1\ 1\ 1$$

Quantum Error Correction

It is possible to reuse **redundancy** in **quantum error correction**. However, there are some complications:

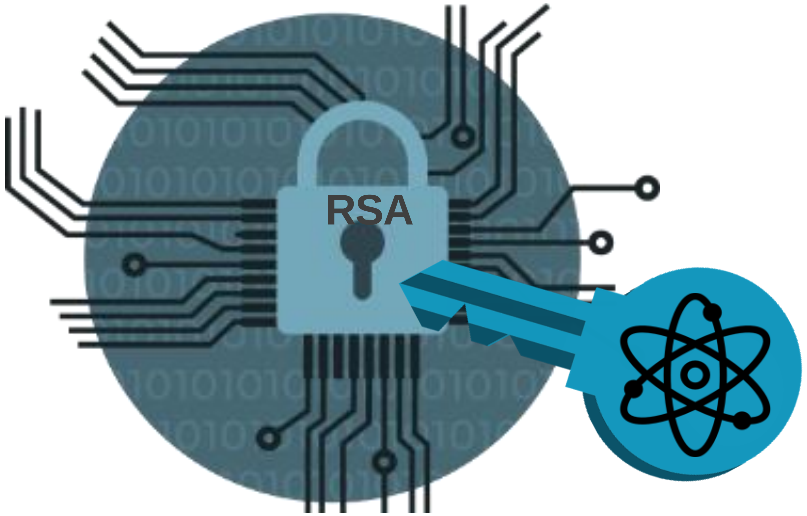
- **No-cloning Theorem**

- Qubits are susceptible to both bit-flips (**X-errors**) and phase-flips (**Z-errors**). (Classically, only bit-flip errors)

- **Measuring affects the quantum state.** Detecting errors must not compromise encoded information

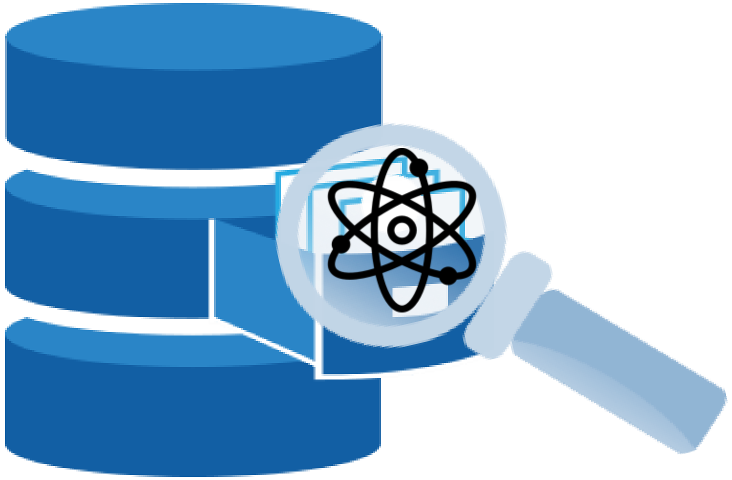
Cryptography

Shor’s Algorithm
Exponential Speedup



Optimization

Grover’s Algorithm
Quadratic Speedup



Cryptography

Shor's Algorithm

Exponential Speedup

Optimization

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Quadratic Speedup

These algorithms assume to have **ideal qubits** that are **not subjected to noise and errors**

Intro to QEC

Cryptography

Shor's Algorithm
Exponential Speedup

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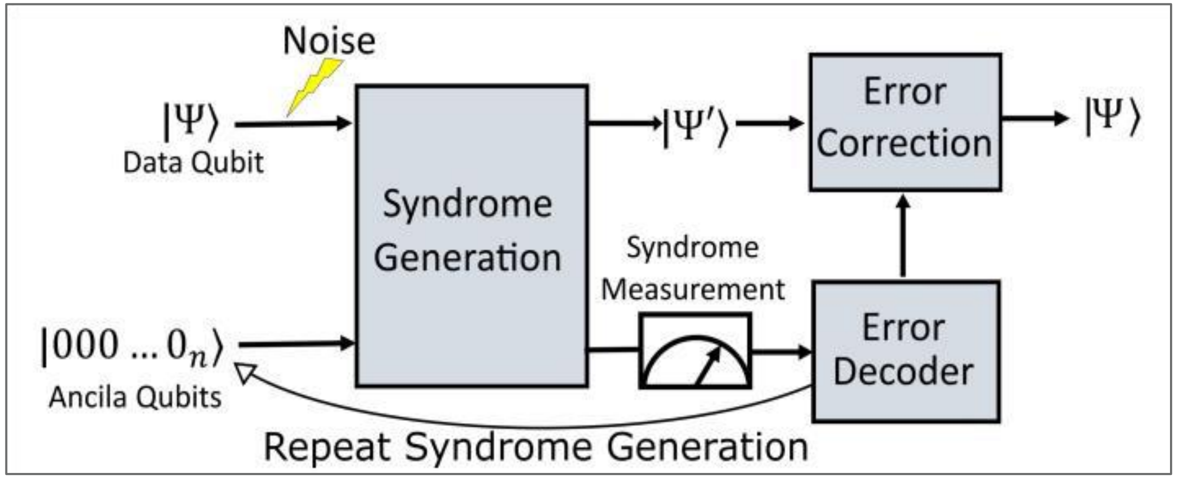
Quadratic Speedup



- Require **error corrected (fault-tolerant)** quantum computers with about **1 million or 100 thousands of qubits**

- Will be availabe in **10-20 years**

Quantum Error Correction

It is possible to reuse **redundancy** in **quantum error correction**. However, there are some complications:

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- **Measuring affects the quantum state.** Detecting errors must not compromise encoded information

No Cloning Theorem

No Cloning theorem

It does NOT exist an universal cloning machine which is a unitary transformation such that

$$U|\psi\rangle|\alpha\rangle = |\psi\rangle|\psi\rangle$$

$\forall |\psi\rangle \in \mathcal{H}$ and $|\alpha\rangle \in \mathcal{H}$ fixed

No Cloning theorem: proof

Suppose such universal cloning machine exists and apply it to two states like below

$$U|\psi_1\rangle|\alpha\rangle = |\psi_1\rangle|\psi_1\rangle \qquad U|\psi_2\rangle|\alpha\rangle = |\psi_2\rangle|\psi_2\rangle$$

No Cloning theorem: proof

Suppose such universal cloning machine exists and apply it to two states like below

$$U|\psi_1\rangle|\alpha\rangle = |\psi_1\rangle|\psi_1\rangle \qquad U|\psi_2\rangle|\alpha\rangle = |\psi_2\rangle|\psi_2\rangle$$

Consider a scalar product between the terms of the eqn.s above

$$\langle\alpha|\langle\psi_2|U^{\dagger}U|\psi_1\rangle|\alpha\rangle = \langle\psi_2|\langle\psi_2|\psi_1\rangle|\psi_1\rangle$$

No Cloning theorem: proof

Suppose such universal cloning machine exists and apply it to two states like below

$$U|\psi_1\rangle|\alpha\rangle = |\psi_1\rangle|\psi_1\rangle \qquad U|\psi_2\rangle|\alpha\rangle = |\psi_2\rangle|\psi_2\rangle$$

Consider a scalar product between the terms of the eqn.s above

$$\underbrace{\langle\alpha|\langle\psi_2|U^\dagger U|\psi_1\rangle|\alpha\rangle}_{\langle\alpha|\alpha\rangle\langle\psi_2|\psi_1\rangle} = \underbrace{\langle\psi_2|\langle\psi_2|\psi_1\rangle|\psi_1\rangle}_{|\langle\psi_2|\psi_1\rangle|^2}$$

$$\underbrace{\langle\alpha|\alpha\rangle\langle\psi_2|\psi_1\rangle}_{} = \underbrace{|\langle\psi_2|\psi_1\rangle|^2}_{}$$

$$\boxed{\langle\psi_2|\psi_1\rangle = |\langle\psi_2|\psi_1\rangle|^2}$$

No Cloning theorem: proof

Suppose such universal cloning machine exists and apply it to two states like below

$$U|\psi_1\rangle|\alpha\rangle = |\psi_1\rangle|\psi_1\rangle \qquad U|\psi_2\rangle|\alpha\rangle = |\psi_2\rangle|\psi_2\rangle$$

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$$\boxed{\langle\psi_2|\psi_1\rangle = |\langle\psi_2|\psi_1\rangle|^2}$$

Contradiction

(true only for orthogonal states)

$$\begin{cases} \langle\psi_1|\psi_2\rangle = 0 \\ \langle\psi_1|\psi_2\rangle = 1 \end{cases}$$

QEC:
Three qubits repetition codes

Repetition codes

This techniques use **redundancy**, **entanglement** and **syndrome measurements** to **correct** single qubits bit-flip and phase-flip errors which may occur with some probability *p*

$$U_{ERR}^{X} = (1-p)\,I + p\,X$$

$$U_{ERR}^{Z} = (1-p)\,I + p\,Z$$

Bit-flip error code

Encoding

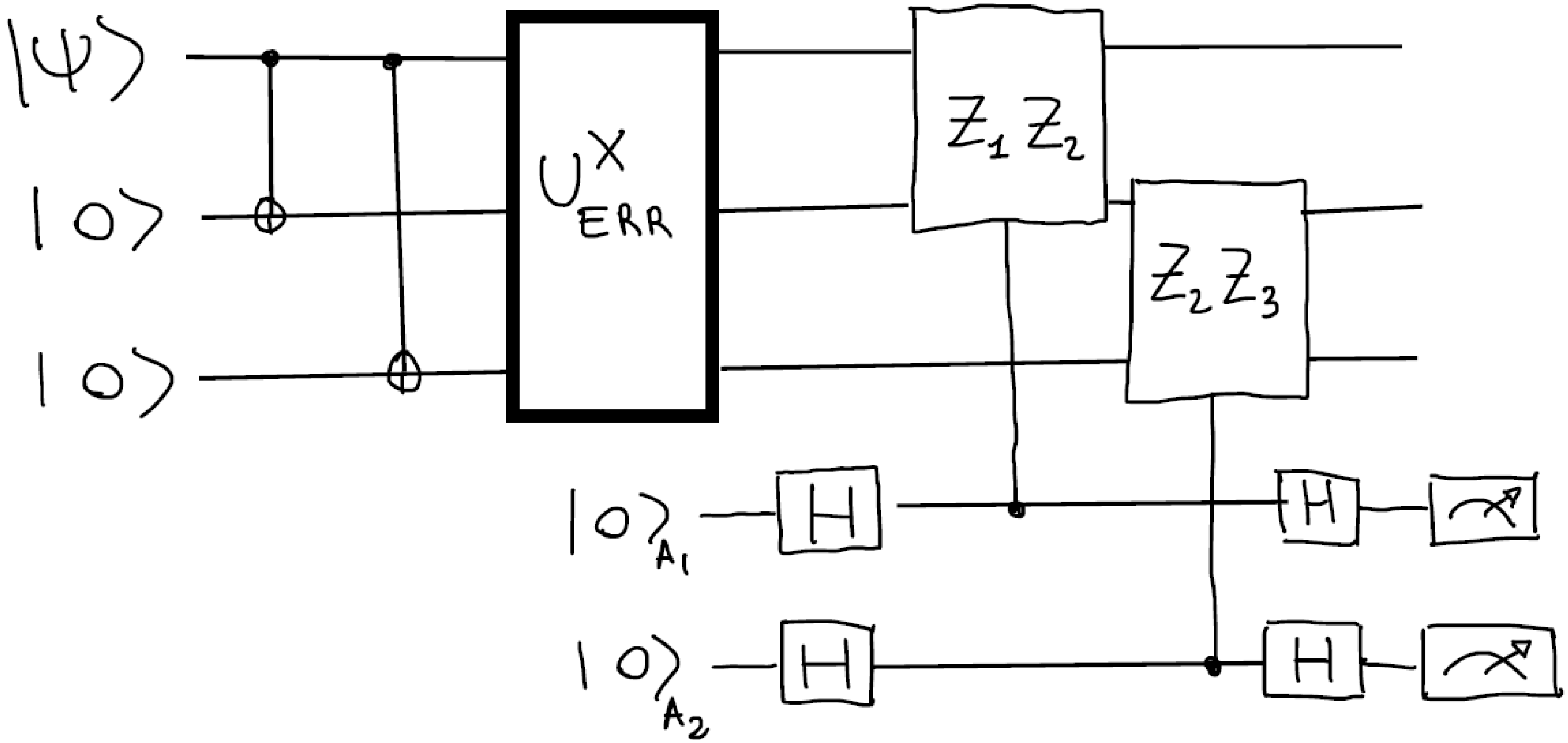
$$|0\rangle \rightarrow |000\rangle$$

$$|1\rangle \rightarrow |111\rangle$$

Syndrome Measurement

$$(Z_1 Z_2), (Z_2 Z_3)$$

Bit-flip errors code

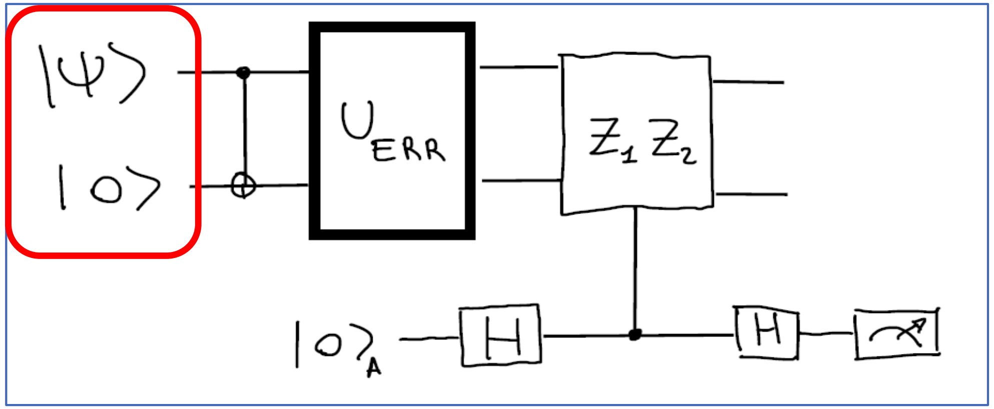


Step by step analysis (two qubit case)

Initial state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\psi\rangle|0\rangle$$

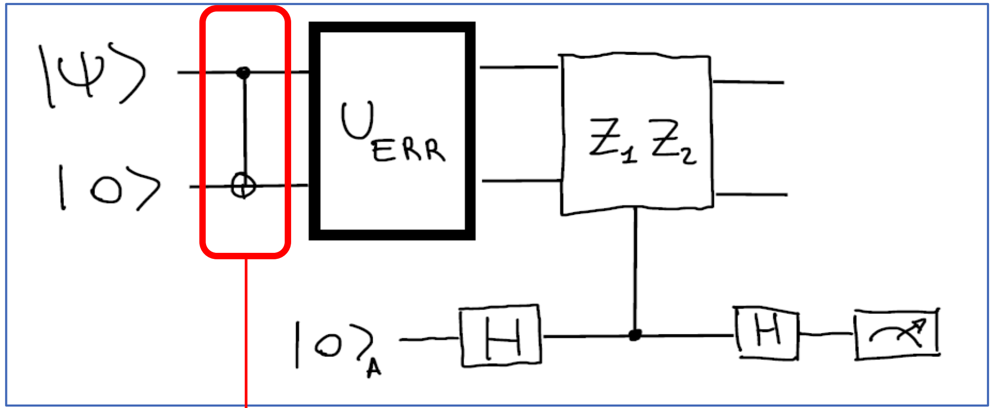


Step by step analysis (two qubit case)

Encoding

$$|0\rangle \longrightarrow |0_L\rangle = |00\rangle$$

$$|1\rangle \longrightarrow |1_L\rangle = |11\rangle$$



Obtained with a Control-X

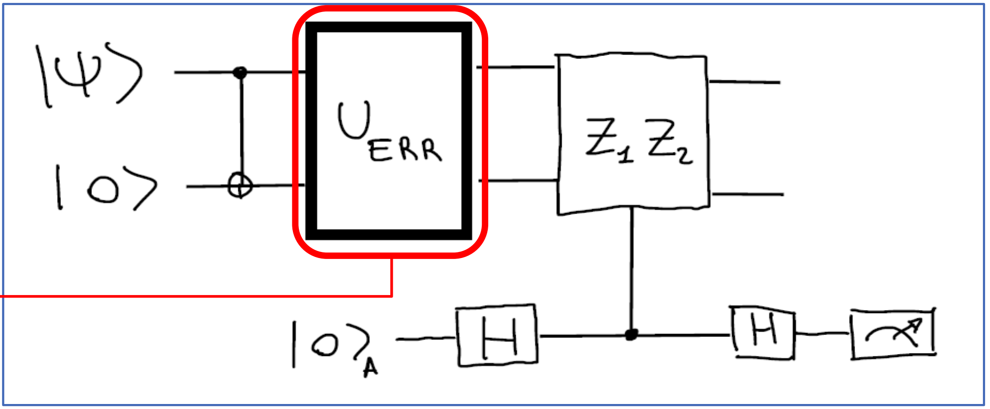
$$\boxed{\begin{aligned} U_{cx}\,|\psi\rangle|0\rangle &= \alpha|00\rangle + \beta|11\rangle \\ &= \alpha|0_L\rangle + \beta|1_L\rangle \end{aligned}}$$

Step by step analysis (two qubit case)

Assume a bit-flip error on the first qubit

$$U_{ERR} = X_1$$

$$\alpha|00\rangle + \beta|11\rangle \xrightarrow{U_{ERR}} \alpha|10\rangle + \beta|01\rangle$$



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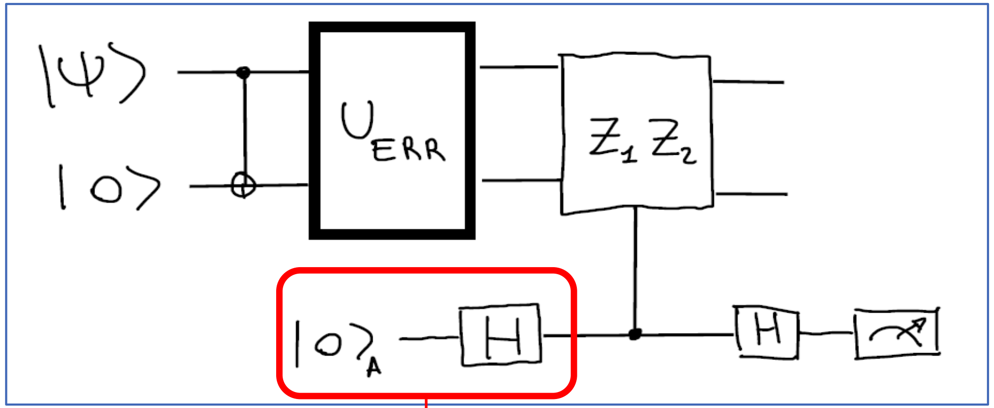

$$|0\rangle_A \xrightarrow{\boxed{H}} \left(\frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}}\right)$$

Bit-flip error codes

Step by step analysis (two qubit case)

Assume a bit-flip error on the first qubit

$$U_{ERR} = X_1$$

$$\alpha|00\rangle + \beta|11\rangle \xrightarrow{U_{ERR}} \alpha|10\rangle + \beta|01\rangle$$

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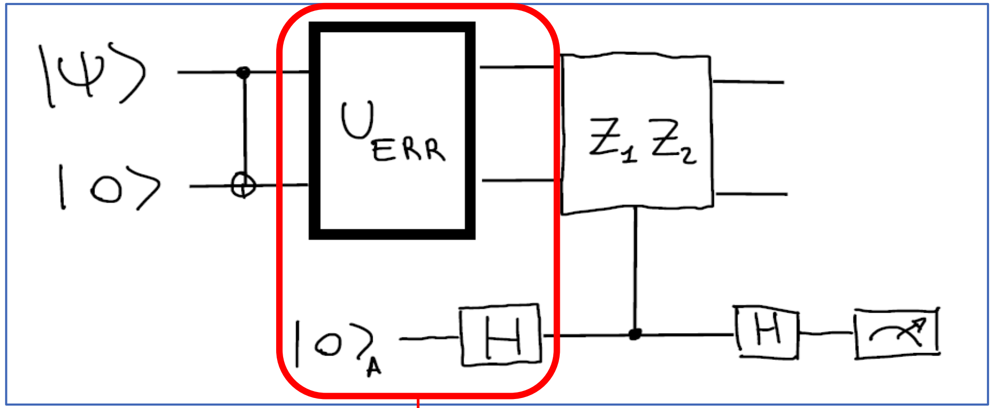


$$\left(\alpha|10\rangle + \beta|01\rangle\right)\left(\frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}}\right)$$

Step by step analysis (two qubit case)

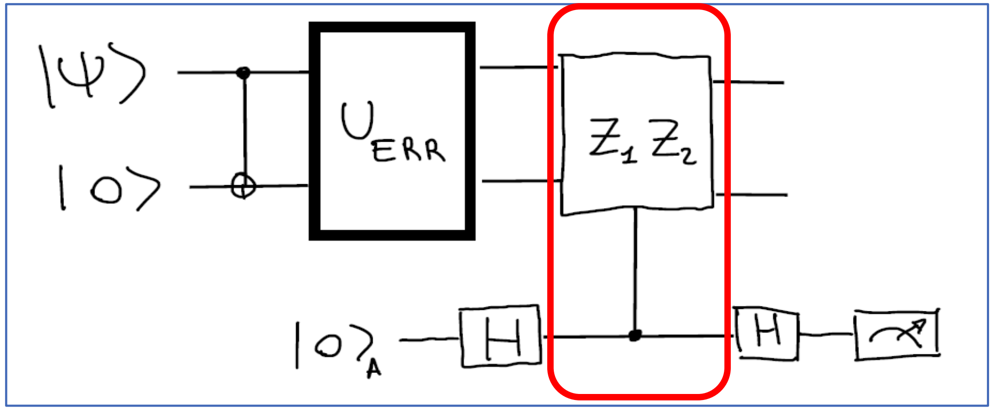
Control gate for syndrome measurement

$$\left(\alpha|10\rangle + \beta|01\rangle\right)\left(\frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}}\right)$$

$$\Big\downarrow C\text{-}Z_1Z_2$$

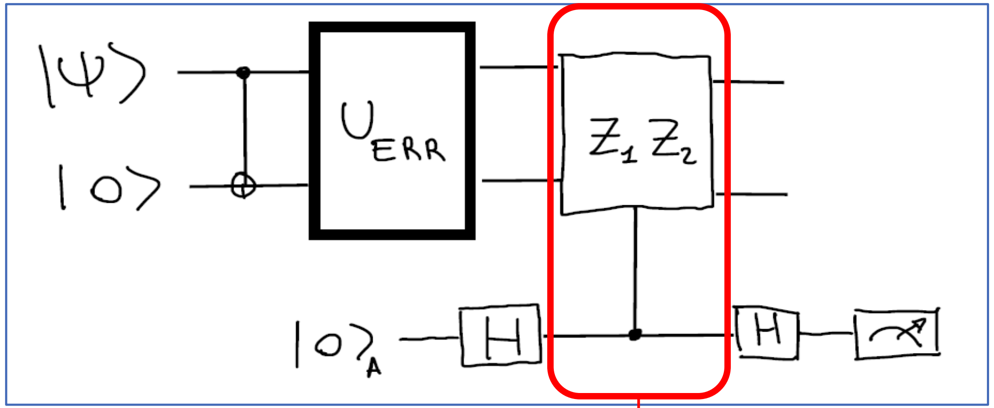
$$|0\rangle_A\left(\frac{\alpha|10\rangle + \beta|01\rangle}{\sqrt{2}}\right) - |1\rangle_A\left(\frac{\alpha|10\rangle + \beta|01\rangle}{\sqrt{2}}\right)$$

$$Z|0\rangle = |0\rangle$$
$$Z|1\rangle = -|1\rangle$$



Step by step analysis (two qubit case)

Control gate for syndrome measurement



$$\left(\alpha|10\rangle+\beta|01\rangle\right)\left(\frac{|0\rangle_A+|1\rangle_A}{\sqrt{2}}\right)$$

$$\Big\downarrow\ C\text{-}Z_1Z_2$$

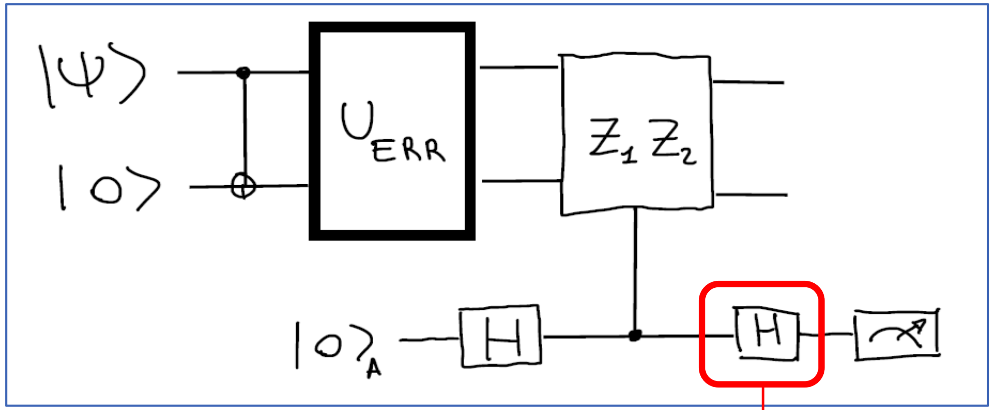
$$Z|0\rangle=|0\rangle$$
$$Z|1\rangle=-|1\rangle$$

$$|0\rangle_A\left(\frac{\alpha|10\rangle+\beta|01\rangle}{\sqrt{2}}\right)-|1\rangle_A\left(\frac{\alpha|10\rangle+\beta|01\rangle}{\sqrt{2}}\right)=\boxed{\alpha|10\rangle+\beta|01\rangle\left(\frac{|0\rangle_A-|1\rangle_A}{\sqrt{2}}\right)}$$

Bit-flip error codes

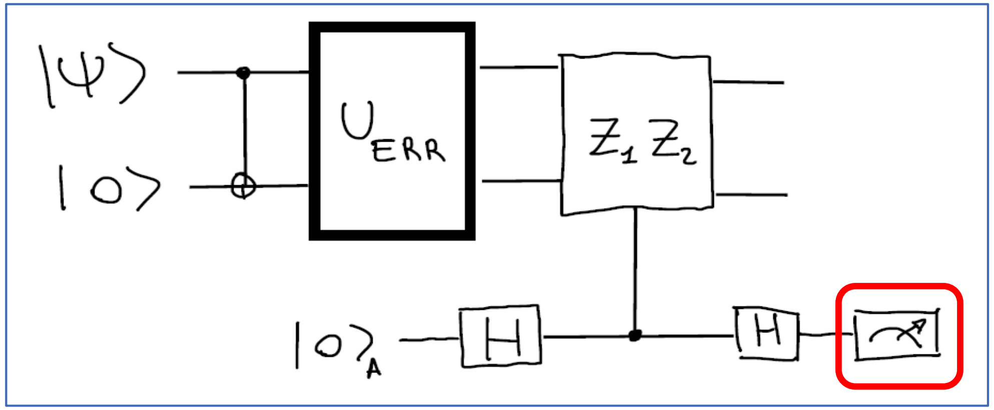
Step by step analysis (two qubit case)

$$\alpha|10\rangle+\beta|01\rangle\left(\frac{|0\rangle_A-|1\rangle_A}{\sqrt{2}}\right)\xrightarrow{H_A}\boxed{\left(\alpha|10\rangle+\beta|01\rangle\right)|1\rangle_A}$$



Step by step analysis (two qubit case)

$$\alpha|10\rangle + \beta|01\rangle \left(\frac{|0\rangle_A - |1\rangle_A}{\sqrt{2}}\right) \xrightarrow{H_A} \boxed{\left(\alpha|10\rangle + \beta|01\rangle\right)|1\rangle_A}$$

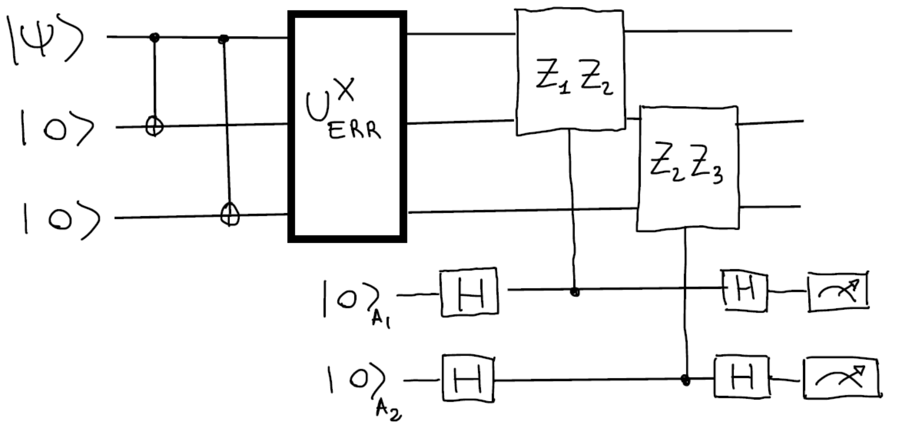


Measuring the state of the ancillary qubit in $|1\rangle_A$ will reveal that a bit-flip error has occurred

Step by step analysis (two qubit case)

$$\alpha|10\rangle+\beta|01\rangle\left(\frac{|0\rangle_A-|1\rangle_A}{\sqrt{2}}\right)\xrightarrow{H_A}\boxed{\left(\alpha|10\rangle+\beta|01\rangle\right)|1\rangle_A}$$

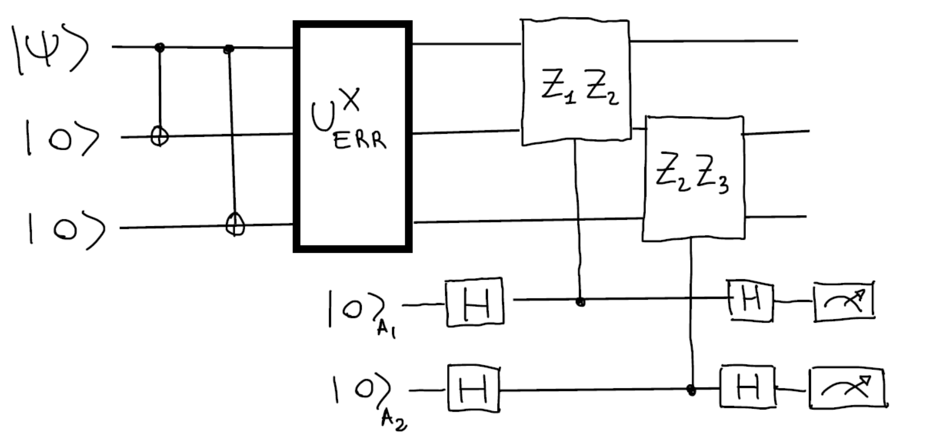
Measuring the state of the ancillary qubit in $|1\rangle_A$ will reveal that a bit-flip error has occurred



We need 3 qubits in the encoding to perfectly locate where bit-flip error has occurred

Step by step analysis (two qubit case)

	$(Z_1 Z_2)$	$(Z_2 Z_3)$	
outcome	$+1$	$+1$	error on none qubits
outcome	-1	$+1$	error on first qubit
outcome	$+1$	-1	error on third qubit
outcome	-1	-1	error on second qubit

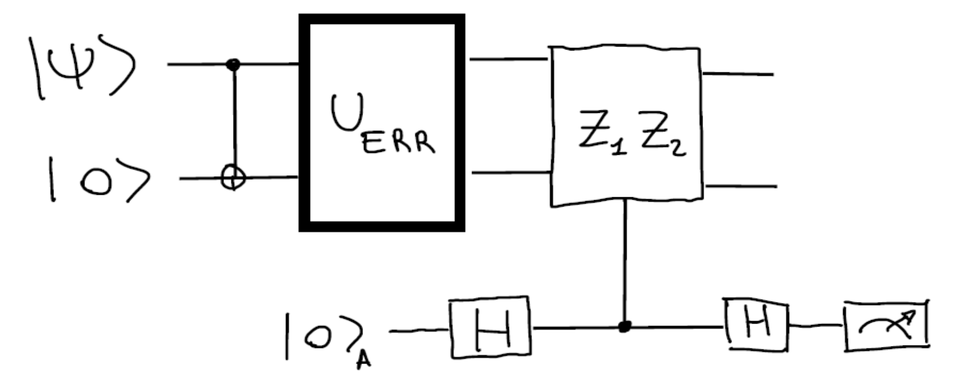


We need 3 qubits in the encoding to perfectly locate where bit-flip error has occurred

Step by step analysis (two qubit case)

Assume a no error on the first qubit

$$\boxed{U_{ERR} = I}$$



$$\alpha|00\rangle + \beta|11\rangle \xrightarrow{U_{ERR}} \alpha|00\rangle + \beta|11\rangle$$

$$\left(\alpha|00\rangle + \beta|11\rangle\right)\left(\frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}}\right) \xrightarrow{C\text{-}Z_1Z_2} \left(\alpha|00\rangle + \beta|11\rangle\right)\left(\frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}}\right) \xrightarrow{H_A}$$

$$\xrightarrow{H_A} \left(\alpha|00\rangle + \beta|11\rangle\right)|0\rangle_A$$

Measuring the state of the ancillary qubit in 0 reveals that no error has occurred

Three qubits repetition codes

Phase-flip errors code

Encoding

$$|0\rangle \to |{+}{+}{+}\rangle$$

$$|1\rangle \to |{-}{-}{-}\rangle$$

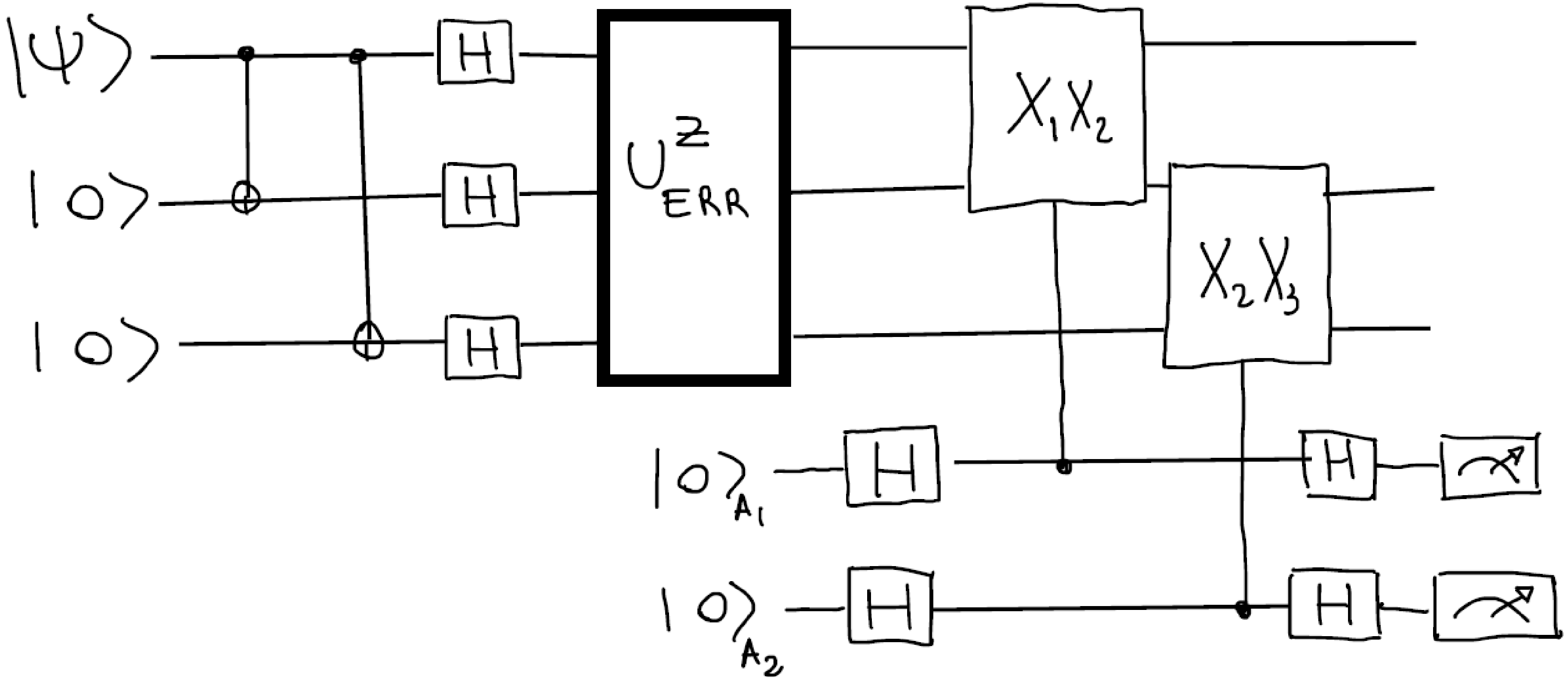
$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|-\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

Syndrome Measurement

$$(X_1 X_2), (X_2 X_3)$$

Phase-flip errors code

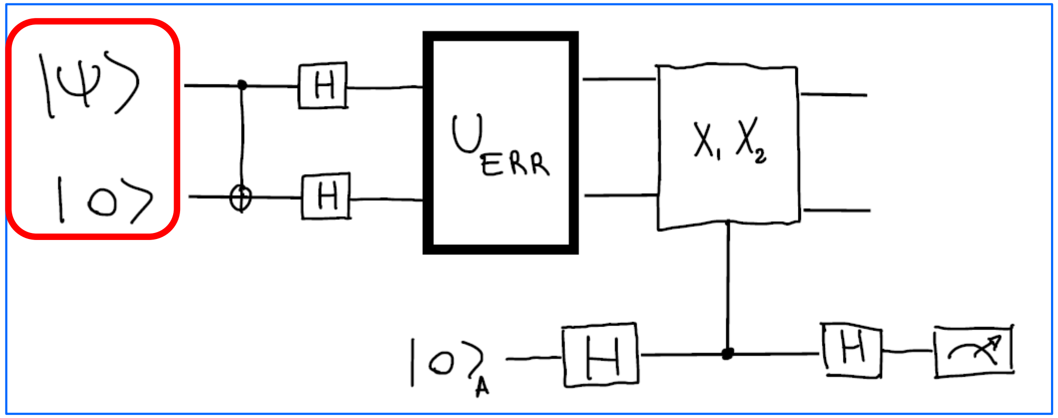


Step by step analysis (two qubit case)

Initial state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\psi\rangle|0\rangle$$



Step by step analysis (two qubit case)

Encoding

$$|0\rangle \rightarrow |{+}\,{+}\rangle$$

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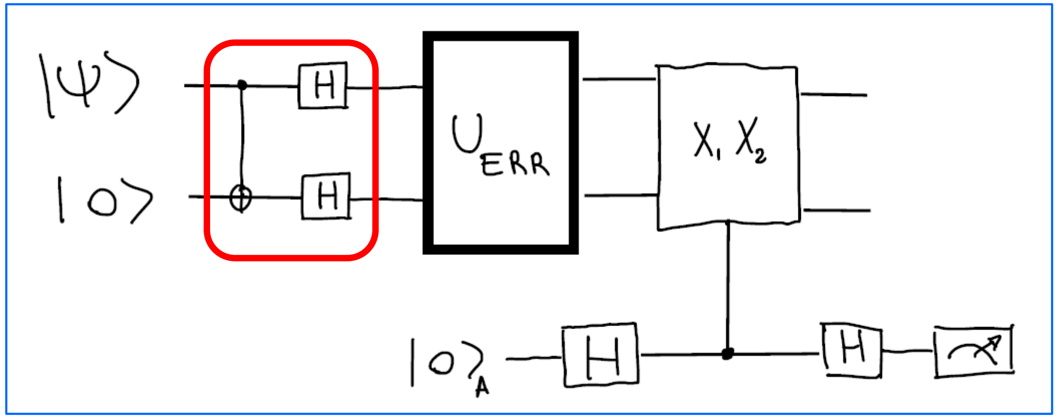


Obtained with a Control-X and Hadamards

$$\left(\alpha|0\rangle + \beta|1\rangle\right)|0\rangle \xrightarrow{U_{CX}} \alpha|00\rangle + \beta|11\rangle \xrightarrow{H^{\otimes 2}} \alpha\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle+|1\rangle}{\sqrt{2}}\right) + \beta\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) =$$

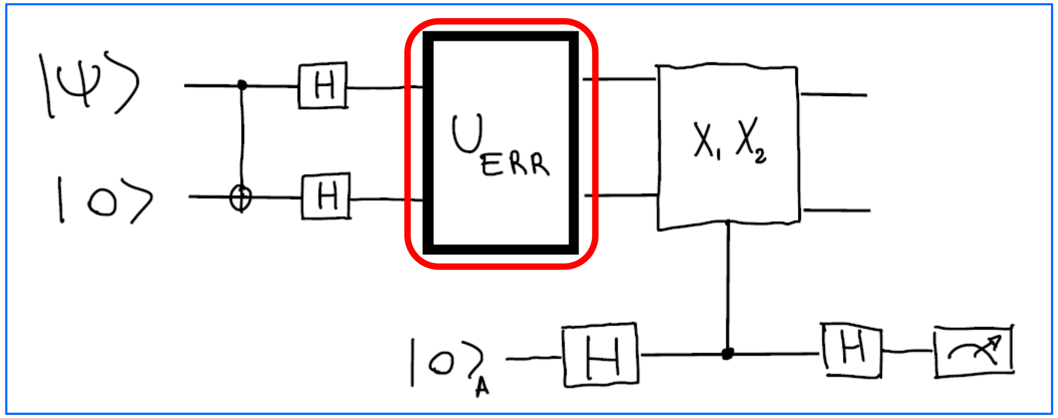
$$\boxed{= \alpha|{+}\,{+}\rangle + \beta|{-}\,{-}\rangle}$$

Step by step analysis (two qubit case)

Assume a phase-flip error on the first qubit

$$U_{ERR} = Z_1$$

$$\alpha|++\rangle + \beta|--\rangle \xrightarrow{U_{ERR}} \alpha|-+\rangle + \beta|+-\rangle$$



Step by step analysis (two qubit case)

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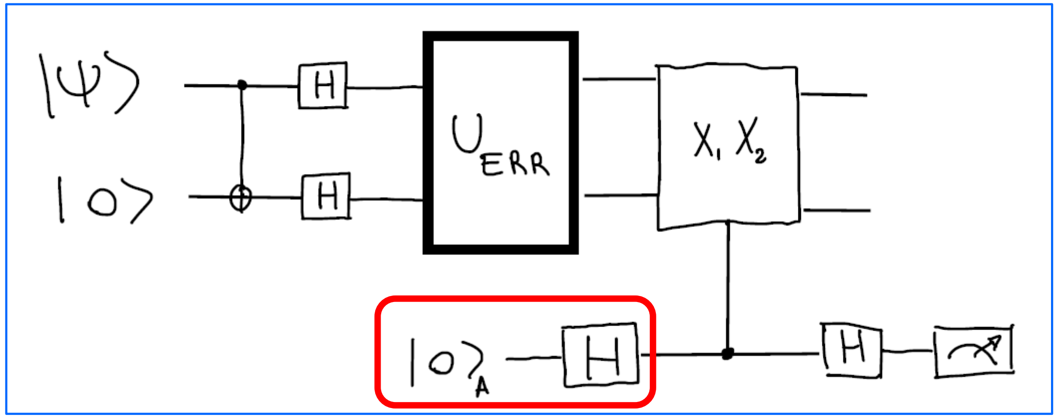
$$|0\rangle_A \xrightarrow{\boxed{H}} \left(\frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}}\right)$$



Phase-flip error codes

Step by step analysis (two qubit case)

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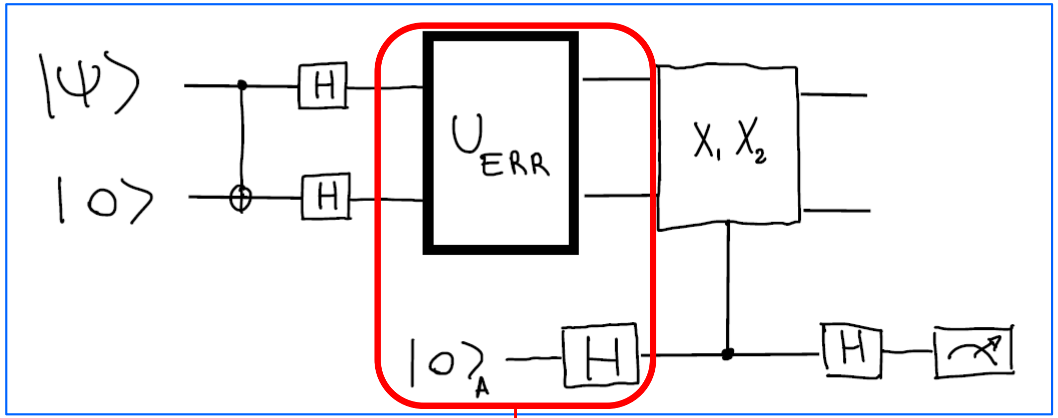

$$\left(\alpha|-+\rangle + \beta|+-\rangle\right)\left(\frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}}\right)$$

Step by step analysis (two qubit case)

Control gate for syndrome measurement

$$\left(\alpha|-+\rangle+\beta|+-\rangle\right)\left(\frac{|0\rangle_A+|1\rangle_A}{\sqrt{2}}\right)$$

$$X|+\rangle=|+\rangle$$
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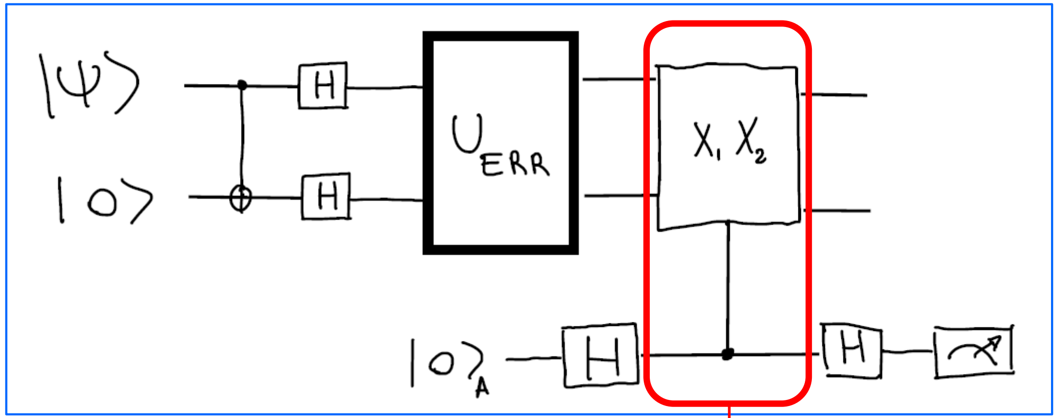


$$\Big\downarrow \; C\text{-}X_1X_2$$

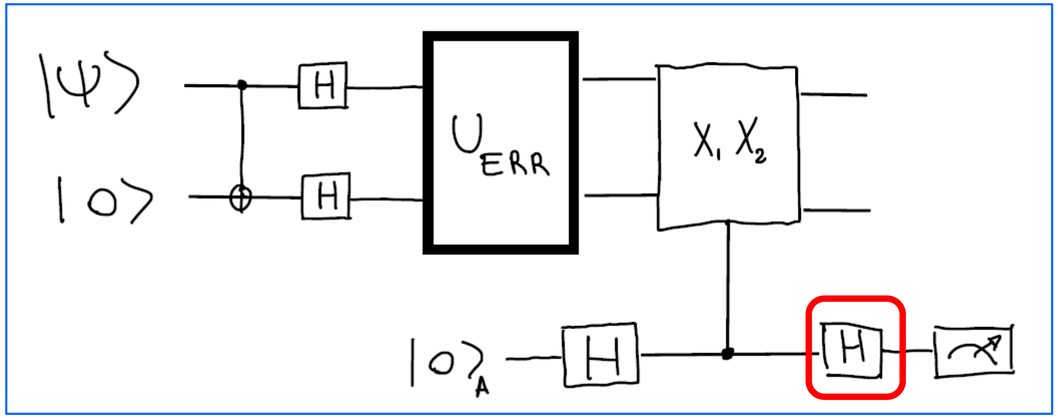
$$|0\rangle_A\left(\frac{\alpha|-+\rangle+\beta|+-\rangle}{\sqrt{2}}\right)-|1\rangle_A\left(\frac{\alpha|-+\rangle+\beta|+-\rangle}{\sqrt{2}}\right)=\boxed{\left(\alpha|-+\rangle+\beta|+-\rangle\right)\left(\frac{|0\rangle_A-|1\rangle_A}{\sqrt{2}}\right)}$$

Step by step analysis (two qubit case)

$$\left(\alpha|-+\rangle + \beta|+-\rangle\right)\left(\frac{|0\rangle_A - |1\rangle_A}{\sqrt{2}}\right)$$

$$\Big\downarrow H$$

$$\boxed{\left(\alpha|-+\rangle + \beta|+-\rangle\right)|1\rangle_A}$$



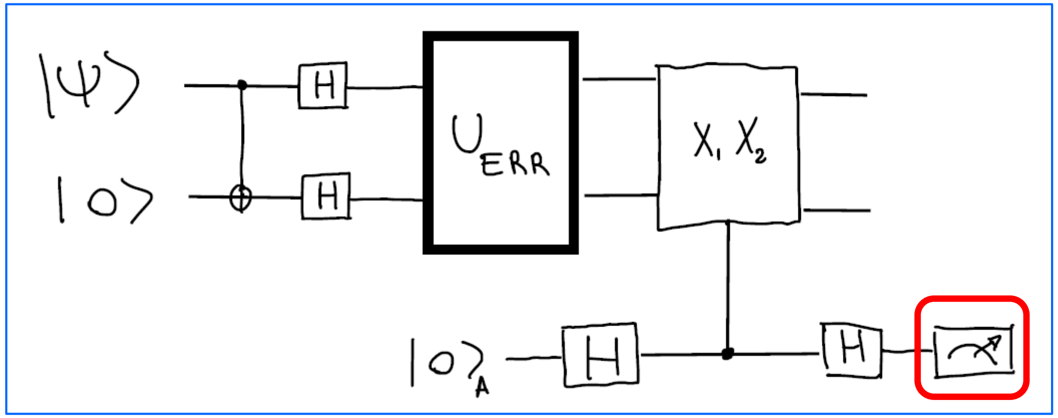
Step by step analysis (two qubit case)

$$\left(\alpha|-+\rangle + \beta|+-\rangle\right)\left(\frac{|0\rangle_A - |1\rangle_A}{\sqrt{2}}\right)$$

$$\Big\downarrow I_A$$

$$\boxed{\left(\alpha|-+\rangle + \beta|+-\rangle\right)|1\rangle_A}$$

Measuring the state of the ancillary qubit in $|1\rangle_A$ will reveal that a phase-flip error has occurred



Phase-flip error codes

Step by step analysis (two qubit case)

$$\left(\alpha|-+\rangle+\beta|+-\rangle\right)\left(\frac{|0\rangle_A-|1\rangle_A}{\sqrt{2}}\right)$$

$$\downarrow$$

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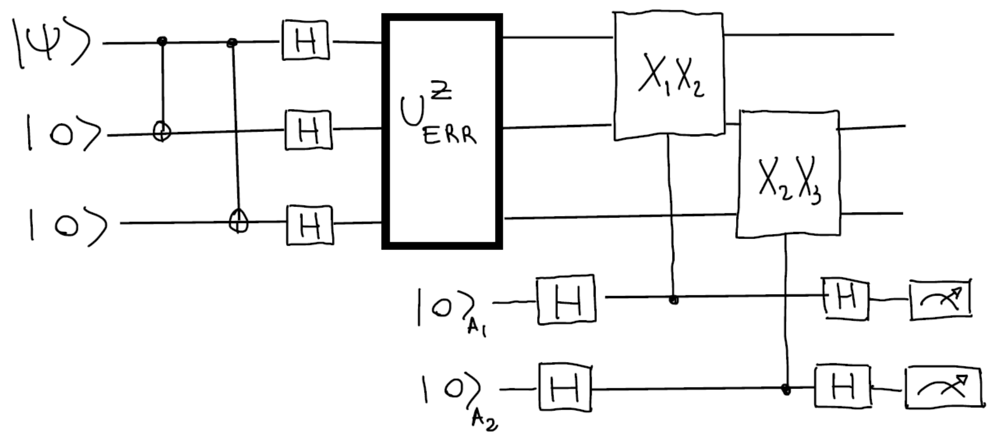

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We need 3 qubits in the encoding to perfectly locate where phase-flip error has occurred

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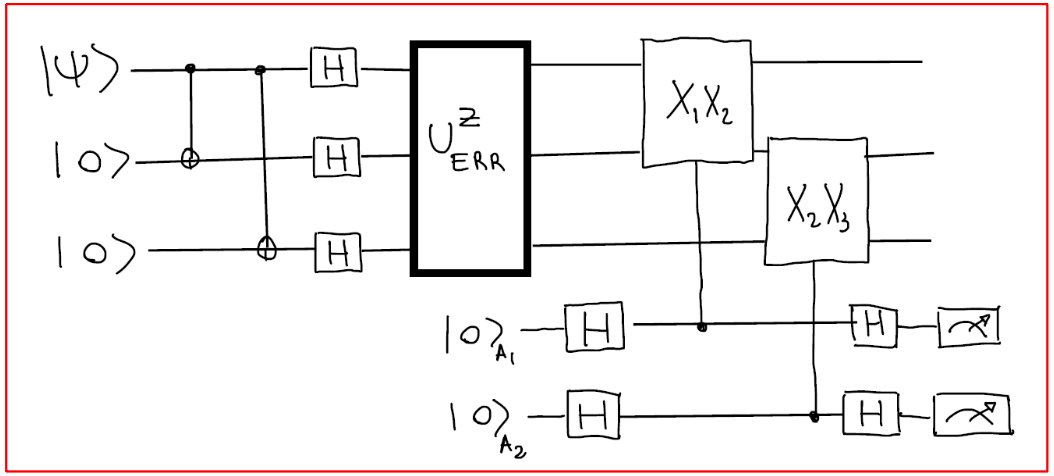

We need 3 qubits in the encoding to perfectly locate where phase-flip error has occurred

Shor Code

Nine qubit repetition code

Uses nine qubits to correct both bit-flip and phase-flip errors

Encoding

Encoding 1

$|0\rangle \to |+++\rangle$

$|1\rangle \to |---\rangle$

then $\rightarrow$

Encoding 2

$|0\rangle \to |000\rangle$

$|1\rangle \to |111\rangle$

Nine qubit repetition code

Uses nine qubits to correct both bit-flip and phase-flip errors

Encoding

Encoding 1

$|0\rangle \to |{+}{+}{+}\rangle$

$|1\rangle \to |{-}{-}{-}\rangle$

then

Encoding 2

$|0\rangle \to |000\rangle$

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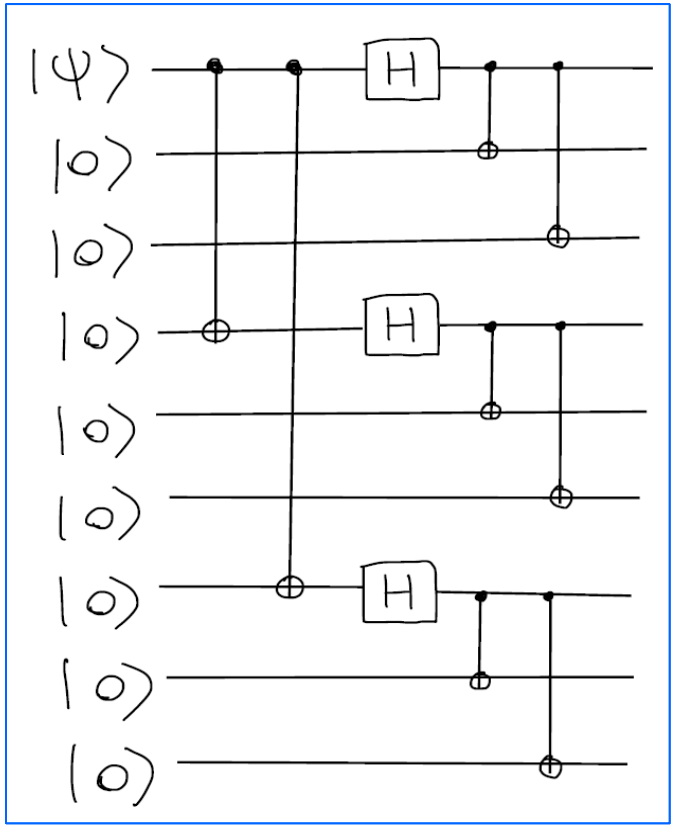

Nine qubit repetition code

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then →

Encoding 2

$|0\rangle \rightarrow |000\rangle$

$|1\rangle \rightarrow |111\rangle$

Syndrome Measurement

$(X_1X_2X_3)(X_4X_5X_6),\ (X_4X_5X_6)(X_7X_8X_9)$

$(Z_1Z_2),\ (Z_2Z_3),\ (Z_4Z_5),\ (Z_5Z_6),\ (Z_7Z_8),\ (Z_8Z_9)$

Quantum Computing @ CINECA

CINECA: Italian HPC center

CINECA Quantum Computing Lab:

- Research with Universities, Industries and QC startups

- Internship programs, Courses and Conference (HPCQC)

https://www.quantumcomputinglab.cineca.it

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